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CHARLOTTE A. SCOTT, A. B. COBLE

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Maps of Twelve Countries with Five Sides with a Group of Order 120 Containing an Ikosahedral Subgroup.

By H. R. Brahana and A. B. Coble.

Introduction.

The figure of the dodecahedron projected from the center upon the surface of the circumscribed sphere yields a regular map of 12 five-sided regions three of which meet at each of 20 corners. This map is transformed into itself by the sixty rotations of the sphere which form an ikosahedral group, g_{60}. It is also transformed into itself by the reflection in the center, an operation of period two which is permutable with all the rotations of g_{60} and which with g_{60} generates a group, G_{120}, of 120 representations of the map upon itself.

In § 1 there is described a regular map of 12 five-sided regions four of which meet at each of 15 corners. The map arises from a division of the projective plane and is transformed into itself by a group which arises from a ternary Cremona group of order 120 which is isomorphic with the permutation group of five things. This $G_{5!}$ is not abstractly identical with the G_{120} of the dodecahedron map but has in common with it the property of containing an ikosahedral subgroup g_{60}. This raises the question as to how many maps of 12 five-sided regions can admit a group of order 120 containing an ikosahedral g_{60}.

It appears in § 2 that there are two abstractly different groups, G_{120} and $G_{5!}$, of this sort. When represented on twelve letters, the group G_{120} gives rise to two distinct substitution groups Γ_{120} and Γ'_{120}, the group $G_{5!}$ to but one substitution group $\Gamma_{5!}$. The method of construction of maps attached to these groups is explained.

In §§ 3, 4, 5 respectively the two maps determined by $\Gamma_{5!}$, the two maps determined by Γ_{120}, and the four maps determined by Γ'_{120} are derived. In § 6 the nature of these eight maps is discussed, and certain of their contacts exhibited.

§ 1

The Map Attached to a Cremona Group Determined by Four Points in a Plane.

Two sets of n ordered points in a plane $p_1, \cdots, p_n$; $q_1, \cdots, q_n$ have been called *congruent* * under a Cremona transformation if the two sets of ρ F-points of the transformation are found at $\rho \geqq n$ of the pairs p_iq_i and if the remaining $n - \rho$ pairs are corresponding points of the transformation. In order to compare two such congruent sets it is important to find geometric objects attached to the one set which pass under the Cremona transformation into the similar geometric objects attached to the other set. For example, if n is not too large the linear system of cubic curves on $p_1, \cdots, p_n$, passes into the linear system of cubic curves on $q_1, \cdots, q_n$. However if $n > 9$ and the points are in general position these curves do not exist. Another type of object defined by the set P_n^2 of points $p_1, \cdots, p_n$ is the aggregate of P-curves attached to the set. This aggregate is defined as follows. Under Cremona transformation the directions about an F-point p_i pass into the points of a rational curve—a P-curve (principal curve) of the transformation—which is uniquely defined by its multiplicities at the points q through which it passes. The aggregate of such P-curves defined by the set Q_n^2 will be called the P-curves of Q_n^2. Thus the P-curves of the set P_n^2 consist of the directions at each of the points, the lines joining any two points, the conics on any five of the points, etc. Under Cremona transformation from P_n^2 to Q_n^2 the P-curves of P_n^2 pass into the P-curves of the congruent set Q_n^2. If $n < 9$ there is a finite number of P-curves attached to P_n^2; if $n \geqq 9$ an infinite number. In any case the aggregate is discontinuous and defines a discontinuous division of the plane. We examine this division for the simplest case, $n = 4$. The P-curves are the four sets of directions at each of the four points and the six lines joining two of the four points. We exhibit the set of directions at a point p by drawing a circle about the point p and taking a pair of opposite points of this circle as the representative of a direction. The map then appears as in Fig. I.

We observe that each of the 12 countries $a, \cdots, f, \alpha, \cdots, \zeta$ is five-sided; three of the sides being parts of the lines $\overline{p_ip_j}$ and two being parts of the directions at two of the points p_i. At every one of the 15 corners four countries come together; for example at the corner on the edge $a\epsilon$ which is at p_4 the countries $\epsilon ac\zeta$ come together. A sketch of this map spread out on a plane with the edges on the border paired in the usual way is given in Fig. II.

* Coble, "Point Sets and Cremona Groups (II)," *Trans. Amer. Math. Soc.*, Vol. 17 (1916), p. 348.

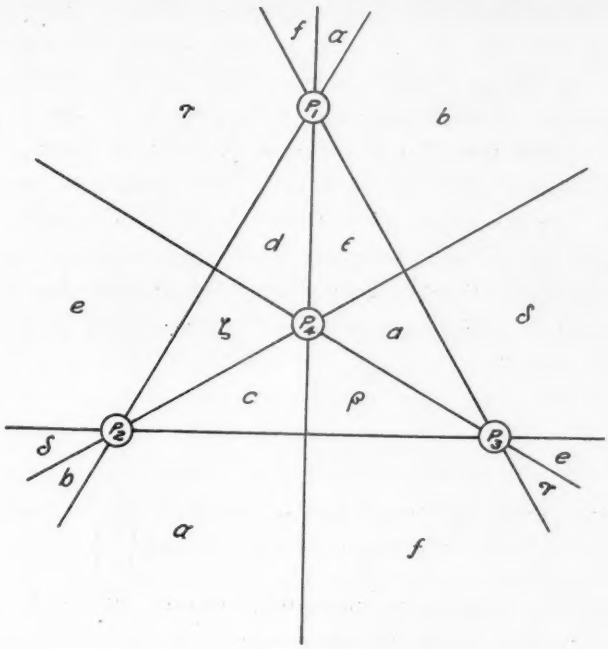
f
α
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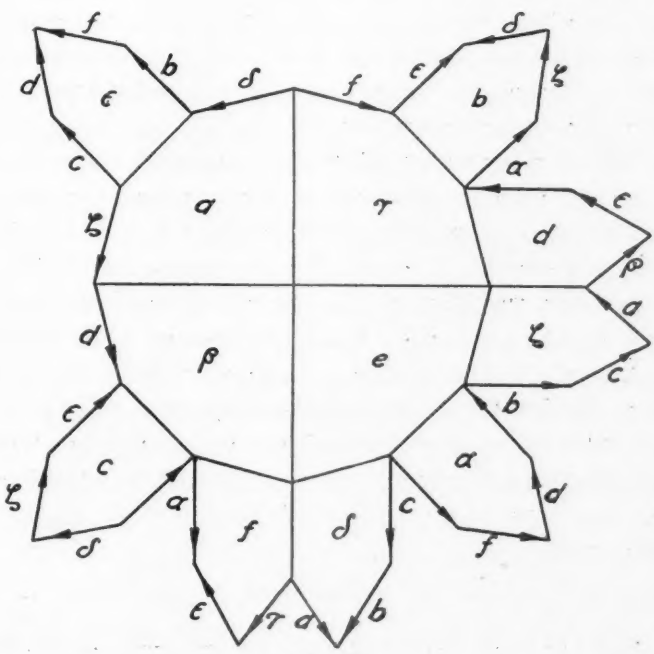
FIG. I.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δ
c
d
δ
f
ε
τ
d
b

FIG. II.

That this map admits a $\Gamma_{5!}$ may be shown as follows. Denote by 1, 2, 3, 4 the pencils of lines on $p_1, \cdots, p_4$ respectively and by 5 the pencil of conics on $p_1, \cdots, p_4$. Then the collineation group, g_{24}, which permutes $p_1, \cdots, p_4$ in all possible ways will permute 1, 2, 3, 4 in all possible ways. The quadratic involution with F-points at p_1, p_2, p_3 and fixed point p_4 interchanges 4 and 5, and leaves 1, 2, 3 each unaltered. This involution, together with g_{24} generates a Cremona group all of whose elements are quadratic except the collineations in g_{24}. The group obviously is isomorphic with the permutation $G_{5!}$ of five things. Parallel generating permutations are $(15234) = (a)(bcdef)(\alpha)(\beta\gamma\delta\epsilon\zeta)$, $(45) = (a\zeta)(f\alpha)(b\delta)(d\beta)(c\epsilon)(e\gamma)$.

§ 2

Substitution Groups of Order 120 on Twelve Letters. Construction of the Map for a Given Group.

There are two types of abstract groups of order 120 with an ikosahedral g_{60}. The first, $G_{5!}$, is the symmetric group on five letters for which g_{60} consists of the even substitutions. We name its elements by the permutations of the five letters written in cycles. The second, G_{120}, consists of the elements of g_{60} and these elements multiplied by an interchangeable element U of period two. We name its elements by the even permutations of five letters or by products of such permutations and U.

G. A. Miller * has shown that these abstract groups give rise to three types of transitive substitution groups on twelve letters. In such a group the elements which leave one letter unaltered constitute a g_{10} which leaves a second letter unaltered. Thus all three types are imprimitive with an imprimitivity system consisting of six pairs of letters. These pairs we take to be $a, \alpha; b, \beta; \cdots; e, \epsilon; f, \zeta$. $G_{5!}$ has but one set of six conjugate g_{10}'s of dihedral type which lie in g_{60}. G_{120} on the other hand has two conjugate sets of g_{10}'s of dihedral type; the one conjugate set of g_{10}'s lies entirely in g_{60} and coincides with the set in $G_{5!}$; the other conjugate set of g_{10}'s arises from the one by multiplying its elements of period two by U. Corresponding to each type of g_{10} there is a transitive substitution group on twelve letters whose generators are:

---

* G. A. Miller, *Quarterly Journal of Math.*, Vol. 28 (1896), pp. 193-231.

$$\Gamma_{5!}\colon\ (15234) = (bcdef)(\beta\gamma\delta\epsilon\zeta);\qquad (45) = (a\zeta)(fa)(b\delta)(d\beta)(c\epsilon)(e\gamma);$$

$$(1)\qquad \Gamma_{120}\colon\ (15234) = (bcdef)(\beta\gamma\delta\epsilon\zeta);\qquad U(15)(23) = (a\epsilon)(ae)(d\zeta)(\delta f)(b\beta)(c\gamma);$$

$$\Gamma'_{120}\colon\ (15234) = (bcdef)(\beta\gamma\delta\epsilon\zeta);\qquad U(15)(23) = (a\epsilon)(ae)(\delta f)(d\zeta);$$

$$\text{where}\quad U = (aa)(b\beta)(c\gamma)(d\delta)(e\epsilon)(f\zeta).$$

We observe that in $\Gamma_{5!}$ there is a second imprimitivity system consisting of the two sets of six letters a, b, c, d, e, f; and $a, \beta, \gamma, \delta, \epsilon, \zeta$; that in the g_{60} of Γ_{120} this second imprimitivity system also exists; whereas in Γ'_{120} this second system exists neither for Γ'_{120} nor for its g_{60}.

A symmetrical map which admits a group can be constructed if there is given an element S of the group which turns an initial country cyclically into itself and also an element T which moves the initial country into an adjacent country. For the corresponding moves for the adjacent country can be obtained by proper transformation of the given elements into new elements, and by a sequence of such moves the entire map is traversed. The element T may always be chosen to be of period two. For if the given T sends a into an adjacent country a_1 it sends a_1 into a country a' adjacent to a_1. The transform of S by T, $T^{-1}ST$, turns a_1 around cyclically and a proper power — say the r-th — of this transform sends a' into a. Thus $T' = T(T^{-1}ST)^r$ sends a into a_1 and a_1 into a. This element T' is of period two. For its square leaves a and a_1 each unaltered, and the edge aa_1 also, and thus effects no change in a and therefore no change in adjacent countries. This involutory element—now taken as T—which interchanges the adjacent countries a, a_1, leaves the edge aa_1 unaltered. It may however be interpreted as a *turn around* the center of the edge aa_1 or as a *turn over* the edge aa_1 according as it does or does not interchange the two corners on this edge. The involutory element T must with S generate a group which is not necessarily of order 120 but must certainly be transitive on the 12 countries. Otherwise the map S, T would not contain 12 countries.

For all three of the groups $\Gamma_{5!}$, Γ_{120}, Γ'_{120} we take S to be the cyclic element of period five which turns a around so that each edge passes into the next:

$$(2)\qquad S = (15234).$$

Then in the case of $\Gamma_{5!}$ and Γ_{120} the countries which bound a are in order $\beta, \gamma, \delta, \epsilon, \zeta$. For otherwise they would be b, c, d, e, f and T would be an involutory element which interchanges a and b and therefore would be in g_{60}.

Then S, T would generate at most g_{60} which is transitive on only 6 letters $a, \cdots, f$. We take therefore a, β as adjacent countries. In the case of Γ'_{120} however either cycle of five letters in S may be taken as the countries adjoining a and we take therefore either a, b as adjacent or a, β as adjacent. Having thus chosen a pair of adjacent countries we select for T an element of period two which contains this pair in a cycle of two letters, interpret T as a turn around or as a turn over the common edge of the two, and proceed to construct the map. Moreover, whenever, as happens in each of the groups under consideration, there are only two elements T and T' of period two containing the adjacent countries a, b or a, β in the same cycle, the interpretation of T as a turn around the common edge implies that T' is a turn over that edge and vice versa. This ensures that the maps obtained by using S and T are the same as those obtained by using S and T'.

Before actually carrying out the construction we observe that the resulting maps are subject to certain restrictions. Since any country can be carried into any other and can be rotated, any corner can be carried into any other corner so that the same number k of countries come in at each vertex. There are $a_2 = 12$ countries each with five edges with each edge in two countries so that there are $a_1 = 30$ edges. If there are a_0 corners with k edges on the corner then since each edge is on two corners

$$a_0 k = 60. \tag{3}$$

On account of the symmetry of the map it will admit rotations of period k around each vertex and therefore the substitutions of the group must contain cycles of k letters. With respect to the cycles in these substitutions we use the theorem of G. A. Miller * to the effect that if a certain cycle is found in m substitutions then the number of letters in these m cycles and in their conjugate cycles is equal to the order of the group. The a_0 conjugate cycles which arise from the vertices must in the case of Γ_{51} and Γ_{120} contain both Greek and Roman letters since adjacent countries are Greek and Roman. Hence these cycles are not in g_{60}. The types of elements in Γ_{51} which are not in g_{60} are as follows:

$$\begin{aligned} (45) &= (a\zeta)(c\epsilon)(d\beta) \,.\, (b\delta)(e\gamma)(fa), \\ (1432) &= (a\delta b\epsilon) \,.\, (d\beta ea) \,.\, (c\gamma)(f\zeta), \\ (12)(345) &= (a\gamma d\beta e\zeta) \,.\, (b\epsilon fac\delta). \end{aligned} \tag{4}$$

Under the operations of the groups these typical elements are transformed

* *Proc. Natl. Acad. of Sciences*, Vol. 9 (1923), pp. 52-4.

into a set of conjugate elements and in these conjugate elements those cycles which are separated by dots are *not* conjugate. The 30 elements conjugate to (1432) furnish 30 cycles conjugate to $(a\delta b\epsilon)$ conformably to Miller's theorem and these 30 cycles furnish sets of four countries at 15 vertices the countries being taken in either order. This is the map of Fig. II for which $k = 4$. The 20 elements conjugate to (12)(345) furnish 20 cycles conjugate to $(b\epsilon fac\delta)$ and these 20 cycles furnish sets of six countries in either order at 10 vertices as in the map of Fig. III for which $k = 6$. Thus in the maps of $\Gamma_{5!}$ $a_0 = 15$ or $a_0 = 10$.

The types of elements in Γ_{120} which are not in g_{60} are, in addition to U, the following:

$$
\begin{aligned}
U(15)(23) &= (b\beta)(c\gamma).\ (a\epsilon)(ea).\ (d\zeta)(f\delta);\\
U(254) &= (a\beta cab\gamma).\ (d\zeta e\delta f\epsilon);\\
U(15234) &= (aa).\ (b\gamma d\epsilon f\beta c\delta e\zeta);\\
U(12453) &= (aa).\ (b\delta f\gamma e\beta d\zeta c\epsilon).
\end{aligned}
\tag{5}
$$

We might expect then to find the maps in Figs. IV, V with $k = 6$, $k = 10$ for which respectively $a_0 = 10$, $a_0 = 6$.

In Γ'_{120} the types of elements which give rise to conjugate sets are, in addition to U, the following:

$$
\begin{aligned}
(15234) &= (fbcde).\ (\zeta\beta\gamma\delta\epsilon);\\
(12453) &= (fcebd).\ (\zeta\gamma\epsilon\beta\delta);\\
(15)(23) &= (ae)(a\epsilon).\ (fd)(\zeta\delta).\ (b\beta)(c\gamma);\\
(254) &= (abc)(a\beta\gamma).\ (dfe)(\delta\zeta\epsilon);\\
U(15234) &= (f\beta c\delta e\zeta b\gamma d\epsilon).\ (aa);\\
U(12453) &= (f\gamma e\beta d\zeta c\epsilon b\delta).\ (aa);\\
U(15)(23) &= (a\epsilon)(ae).\ (f\delta)(\zeta d);\\
U(254) &= (a\beta cab\gamma).\ (\delta f\epsilon d\zeta e).
\end{aligned}
\tag{6}
$$

Here then we might expect to find maps as in Figs. VI, · · ·, IX for which $k = 3, 5, 6, 10$ or $a_0 = 20, 12, 10, 6$.

§ 3

The Two Maps Belonging to $\Gamma_{5!}$.

We select an initial country a bordered as we have seen by $\beta, \gamma, \delta, \epsilon, \zeta$ for which the substitution S is

$$S = (15234) = (a)(bcdef)(a)(\beta\gamma\delta\epsilon\zeta).$$

We attach the involutory element T to the edge $a\beta$ and find that there are two elements of the group of the type (45) in (4) which contain the cycle $(a\beta)$. One of these is

$$T = (12) = (a\beta)(d\zeta)(e\gamma)\,.\,(c\epsilon)(f\delta)(ba).$$

1°. *We interpret T as a turn over the edge $a\beta$* and look for the countries which come in at the corner $a\beta\gamma = v$.

Transforming T by S we get

$$(35) = (a\gamma)(e\beta)(f\delta).\ (ac)(\epsilon b)(\zeta d)$$

which as we expect sends a into γ and is the turn $T_{a\gamma}$ over the edge $a\gamma$. The turn over the other edge of γ at v is the transform of $T = T_{a\beta}$ by $T_{a\gamma}$. But this transform is T again and T sends γ into e, the next country at v. Thus $T = T_{e\gamma}$ and the transform of $T_{a\gamma} = (35)$ by $T_{e\gamma} = (12)$ is $T_{a\gamma}$ itself which sends e into β and closes the cycle $\beta a\gamma e$ at the vertex v. Looking now at Fig. II we begin with bordered a and border β and γ by using $T_{a\beta}$ and $T_{a\gamma}$ respectively. Applying $T = T_{e\gamma}$ to γ we obtain bordered e. The country ϵ is now constructed and bordered by applying S^2 to γ. By applying $T_{a\gamma}$ to ϵ, b is bordered; and by applying T to ϵ and b, the countries c, a are bordered. By using S^{-1}, f is bordered from b; and by using S d is bordered from c. Then $T_{a\gamma}$ sends bordered f into bordered δ; and T sends d into bordered ζ. The map is now complete except for the pairing of the edges on the border. This is accomplished most conveniently by writing the ten corner cycles conjugate to $(\beta a\gamma e)$ and marking these corners on the border. Then the arrow on the edge $a\delta$ of country a is correlated to that on the edge δa of country δ from the marking of the corresponding vertices on this edge. The results are indicated on the figure.

2°. *We interpret T as a turn around the edge a* and look for the countries at the corner $\beta a\gamma = \mathrm{v}$. Transforming $T = T_{a\beta}$ by S we get $T_{a\gamma} = (35)$ as before which turns a into the adjacent country γ. If $S = S_a$ is transformed by $T_{a\gamma}$ the result is $S_\gamma = (13254) = (afbde)(a\zeta\beta\delta\epsilon)$, a rotation of γ in the direction from the edge $a\gamma$ to the next edge of γ at v. If then $T_{a\gamma}$ is transformed by S_γ it becomes the turn around this next edge. The transform is $(24) = (a\delta)(f\gamma)(b\epsilon)\,.\,(e\beta)(c\zeta)(da)$ which sends γ into f so that we label (24) as $T_{\gamma f}$ and mark f as the next country at v. Transforming S_γ by $T_{\gamma f}$ we get $S_f = (13452) = (abdce)(a\beta\delta\gamma\epsilon)$; and transforming $T_{\gamma f}$ by S_γ we get $T_{f\epsilon} = (15) = (b\gamma)(f\epsilon)(da).\ (a\delta)(e\zeta)(c\beta)$ which turns f into ϵ and marks ϵ as the next country to f at v. Transforming S_f by $T_{f\epsilon}$ we have

$S_\epsilon = (12534) = (abfdc)(a\beta\zeta\delta\gamma)$; and transforming $T_{f\epsilon}$ by S_ϵ we get $T_{\epsilon d} = (23) = (d\epsilon)(c\beta)(fa).\ (a\zeta)(b\gamma)(e\delta)$ which turns ϵ into d the next country to ϵ at v. Transforming S_ϵ by $T_{\epsilon d}$ we have $S_d = (13524) = (aebfc)(a\epsilon\beta\zeta\gamma)$; and transforming $T_{\epsilon d}$ by S_d we get $T_{d\beta} = (45) = (a\zeta)(c\epsilon)(d\beta).\ (b\delta)(e\gamma)(fa)$ which turns d into β and completes the cycle $\beta a\gamma f\epsilon d$ of six countries at v.

This cycle of six countries at v is found in

$$\Sigma = (52)(143) = (e\zeta cab\delta).\ (a\beta d\epsilon f\gamma)$$

and the rotation about v is furnished by Σ. If then we carry out Σ on

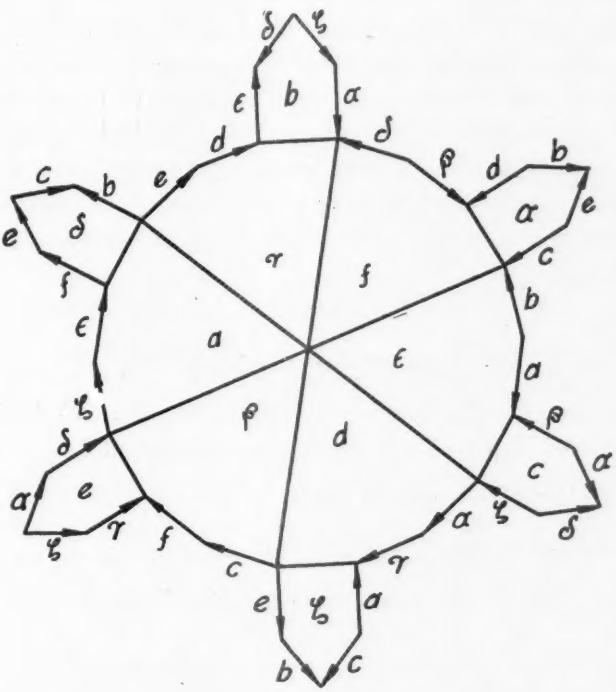

FIG. III.

bounded a we get the six inner countries in Fig. III each bounded. Then S applied to bounded γ furnishes bounded δ and Σ applied to bounded δ completes the map. By marking on the border the vertices which are obtained from the cycle in Σ and its conjugates the directions on the paired edges of the border are determined.

There is another involutory element namely

$$T' = (34) = (e\zeta)(d\gamma)(b\alpha).\ (a\beta)(c\delta)(f\epsilon)$$

which contains the cycle $(a\beta)$ and which might be used with S to construct a map. The maps so obtained would not differ from the two already found. For in the ten elements conjugate to $T = (12)$ there are sixty cycles of two letters, each denoting one of the thirty edges. But these sixty cycles divide into two conjugate sets, a first set of three and a second set of three in each element. In the first map in which $T = (12)$ was used as a turn over the edge $a\beta$ we observe that it is also a turn over the edges $e\gamma$ and $d\zeta$ in the same conjugate set but that it is a turn around the center of the three edges $c\epsilon$, $f\delta$, $b\alpha$ in the other conjugate set; in the second map T is a turn around the first three edges and a turn over the second three edges. This double use of T implies of course a one-sided map as explained in § 6. Thus the map obtained by taking the involutory elements as turns over or around the edges in the first set of conjugate cycles are precisely those obtained by taking the involutory elements as turns around or over respectively the edges in the second set of conjugate cycles. Since in T $(a\beta)$ occurs in the first set of cycles and in T' $(a\beta)$ occurs in the second set of cycles no new maps are obtained.

§ 4

The Two Maps Belonging to Γ_{120}.

The procedure for Γ_{120} is precisely like that for Γ_{51}. We begin with the country a bordered by β, γ, δ, ϵ, ζ which is turned around by

$$S = (15234) = (bcdef)(\beta\gamma\delta\epsilon\zeta)$$

and for the edge $a\beta$ we select the involutory element

$$T = U(14)(23) = (d\delta)(e\epsilon).\ (a\beta)(b\alpha).\ (f\gamma)(c\zeta)$$

and again

1°. *We interpret T as a turn over the edge $a\beta$.*

Exactly as in § 3 we find the countries at the corner $v = \beta a\gamma$ to be the cycle of ten, $\beta a\gamma d\zeta bac\delta f$, which occurs in the substitution

$$\Sigma = U(14352) = (e\epsilon).\ (a\gamma d\zeta bac\delta f\beta).$$

We put the corner v of this cycle at the center of the map as in Fig. IV and by applying Σ to bordered a we border all the countries in the cycle. By

applying S^3 to bordered γ we obtain bordered ϵ; and from ϵ by using Σ^5 we obtaining bordered e. By distributing the six vertices obtained from the cycles of Σ and its conjugates over the border we correlate the edges of the map.

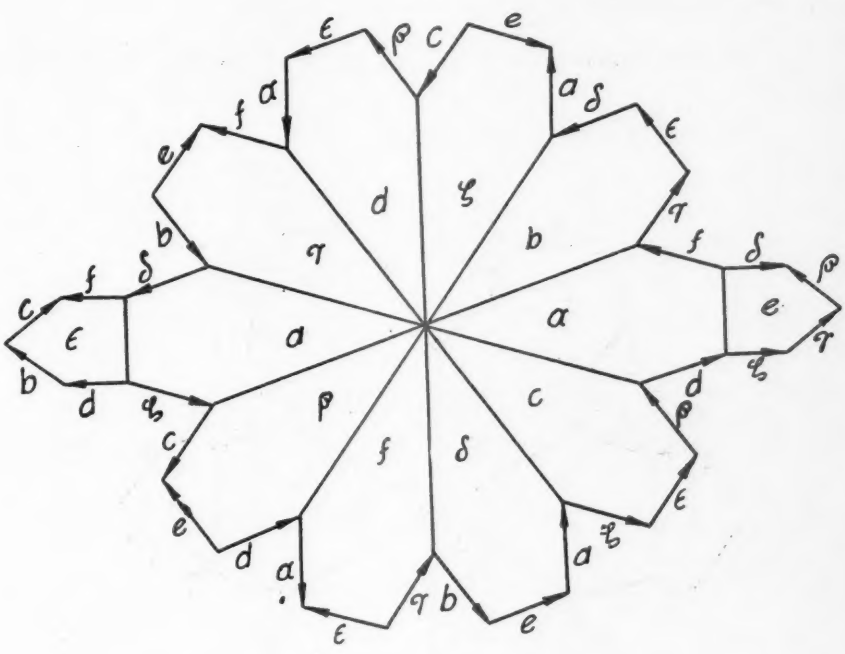

FIG. IV.

2°. *We interpret T as a turn around the edge $a\beta$.* Then as in § 3 we find the cycle of six countries $\beta a\gamma bac$ at the corner $v = \beta a\gamma$, a cycle of the substitution

$$\Sigma' = U(254) = (a\beta cab\gamma).\ (d\zeta e\delta f\epsilon).$$

Putting this cycle at the center as in Fig. V we apply Σ' to bounded a to bound the cycle; apply S^2 to bounded γ to obtain bounded ϵ and apply Σ' to bounded ϵ to complete the map. The ten vertices are found from the conjugates of the first cycle of Σ' and the edges are correlated as usual.

We verify for the same reason as in § 3 that the element $T' = U(42)(31) = (c\gamma)(f\zeta).\ (e\delta)(d\epsilon).\ (ba)(a\beta)$, which also can be interpreted as a turn over or around the edge $(a\beta)$, leads to the same maps, IV and V.

§ 5

The Four Maps Belonging to Γ'_{120}.

We continue the method of §§ 3 and 4. In this case, however, we may border a by b, c, d, e, f or by $\beta, \gamma, \delta, \epsilon, \zeta$.

1°. We first border a by b, c, d, e, f. We consider a to be turned around by

$$S = (15234) = (bcdef)(\beta\gamma\delta\epsilon\zeta),$$

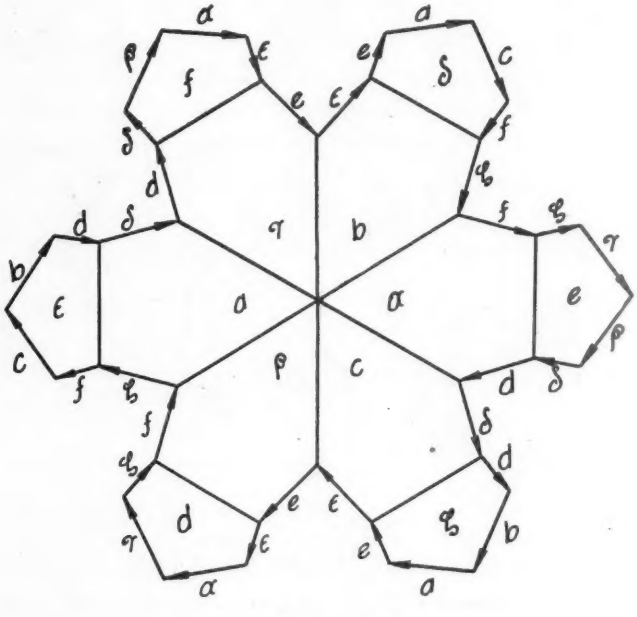

FIG. V.

and for the edge ab we select the element

$$T = (14)(23) = (ab)(a\beta)(cf)(\gamma\zeta)(d\delta)(e\epsilon),$$

which we interpret as a *turn around the edge ab*. The countries at the corner $v = baf$ are simply baf which is a cycle in the substitution of period 3, $\Sigma_1 = (135) = (afb)(a\zeta\beta)(\gamma\epsilon d)(ce\delta)$. Applying Σ_1 to bordered a we border countries b and f. S^2 transforms Σ_1 into $\Sigma_1' = (132) = (acd)(a\gamma\delta)$

$(\epsilon\beta f)(eb\zeta)$ which enables us to border c and d from bordered a (Fig. VI). The inverse of Σ_1 applied to bordered d gives us bordered ϵ; this completes the boundary of ζ also. Σ_1 applied to bordered c gives bordered e, sends bordered ζ into bordered β, bordered d into bordered γ, and finally bordered e into bordered δ. The country a is the only one left; it touches countries

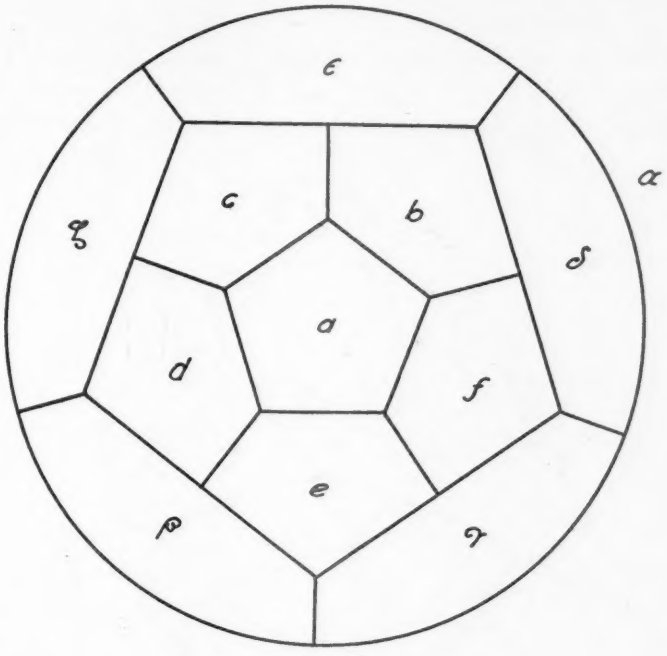

FIG. VI.

β, γ, δ, ϵ, ζ, in that order as we see by applying Σ_1' to bordered δ. The eleven regions first constructed may be represented on a sphere in which case the remainder of the sphere will represent the region a. The map is the dodecahedron.

2°. Next we border a by β, γ, δ, ϵ, ζ. We consider a to be turned around by S and for $a\beta$ we select the element

$$T' = (13)(24) = (a\beta)(ab)(\delta\epsilon)(de)(c\gamma)(f\zeta)$$

which we interpret to be a *turn around the edge* $a\beta$. The countries at the corner $v = \beta a\zeta$ are found to be β, a, ζ, e, c which is one of the cycles of $\Sigma_2 = (12345) = (\beta a\zeta ec)(baf\epsilon\gamma)$. We put v at the center and apply Σ_2

to bound the cycle. Σ_2 transformed by S puts bounded β into bounded f and f into d and then powers of Σ_2 give bounded ϵ, γ, b and a. S transforms bounded γ into bounded δ and completes the map. The vertices may be obtained from the conjugates of Σ_2 and the edges paired as in §§ 3 and 4. It will be observed that the last two maps are on two-sided surfaces, a matter which will be considered further in § 6.

3°. We take a to be bordered by b, c, d, e, f and take S and T as in 1°. Now, however, *we interpret T as a turn over the edge ab.* We find the cycle of ten countries $b, a, f, e, \gamma, \beta, \alpha, \zeta, \epsilon, c$ at the corner $v = baf$, a cycle of $\Sigma_3 = U(12345) = (bafe\gamma\beta\alpha\zeta\epsilon c)(d\delta)$. We put v at the center (Fig. VIII)*

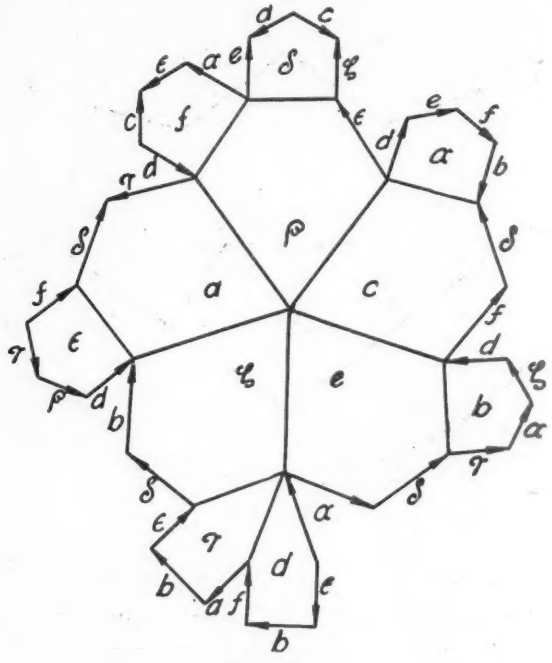

FIG. VII.

and bound the cycle. S applied to bounded c gives bounded d which in turn

* For the sake of convenience we have adopted for maps VIII and IX the following notation:

Figure VIII, $\lambda_1 = U(12345)$, $\lambda_2 = U(12534)$, $\lambda_3 = U(15324)$, $\lambda_4 = U(15243)$, $\lambda_5 = U(14523)$, $\lambda_6 = U(13542)$,

Figure IX, $\lambda_1 = U(135)$, $\lambda_2 = U(254)$, $\lambda_3 = U(132)$, $\lambda_4 = U(354)$, $\lambda_5 = U(142)$, $\lambda_6 = U(235)$, $\lambda_7 = U(125)$, $\lambda_8 = U(145)$, $\lambda_9 = U(134)$, $\lambda_{10} = U(342)$

gives bounded δ with the application of Σ_3^5. Vertices are designated and edges paired as before. The similarity of Figs. V and VIII is only superficial as may be seen by examining the way the edges are paired.

4°. Finally we take the boundary of a, and substitutions S and T' as in 2°. *We interpret T' as a turn over the edge $a\beta$.* $v = \beta a \zeta$ belongs to the cycle $(\beta a \zeta b a f)$ of $\Sigma_4 = U(135) = (\beta a \zeta b a f)(\gamma e d c \epsilon \delta)$. We apply Σ_4 to bound the other five regions of the cycle. By means of β and S^2 we bound δ and then apply Σ_4 to complete the map.

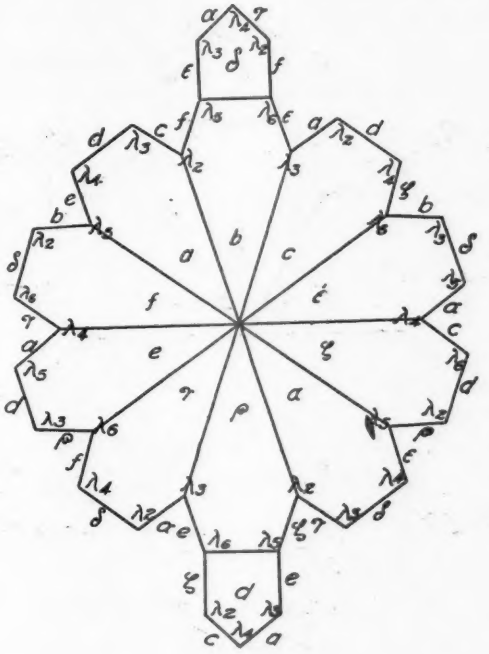

FIG. VIII.

§ 6

The classification of the surfaces on which these various maps lie is of interest. One, the dodecahedron, is familiar, and needs no discussion here. All the others have been represented so far on polygons in the finite part of the plane. These polygons have their sides paired. It is well-known * that

* H. R. Brahana, *Annals of Math.*, Vol. 23, 2nd ser. (1922), pp. 144-168.

such a polygon determines a two-dimensional manifold without singularities which is *two-sided* or *one-sided* according as there does not or does exist a pair of corresponding sides whose members have the same sense on the boundary of the polygon. The polygon in Fig. VII has no such pair. If the map in Fig. VI were represented on a polygon it would have no such pair. Each of the others has at least one pair of corresponding sides whose members have the same sense. Thus, the other six maps are on one-sided surfaces.

The above result is one that may be foreseen by a consideration of the operations we used to construct the maps. The two-sided surfaces get the

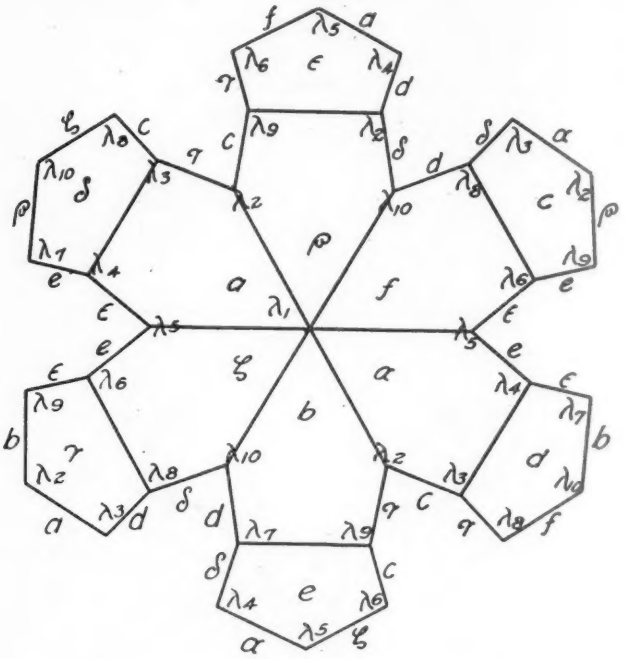

FIG. IX.

name *orientable* from the fact that in any division of the surface into regions (or 2-cells) it is possible to assign a sense, which we may call positive, to each region in such a way that any displacement in two dimensions of a region into any other region carries the positive sense of that first region into the positive sense of the other.* On a one-sided surface this may not be done, for in any division of the surface into sensed polygons it is always possible

* Veblen and Young, *Projective Geometry*, Vol. II, p. 495.

to displace positive a into negative a. In the three preceding sections the operation S determines a sense for a by giving us the cyclic order of the countries which have edges in common with a. When we assign the names to the countries we choose whether $\beta, \gamma, \delta, \epsilon, \zeta$ shall be put down in a clock-wise or a counter-clockwise direction. Then S transformed by T gives the boundary of a neighboring region β. The sense determined by the transform of S depends upon the interpretation we give to T. In the cases where we have interpreted T as a *turn around the center of the line* $a\beta$ the operation T has been a displacement, in the other cases it has not been a displacement. In case T is a displacement we bound the second region so that reading in a clock-wise manner we get one of the cycles of the transform of S (the boundary of a in a clock-wise manner gives a cycle of S). With the other interpretation of T, the boundary in a clock-wise direction of a transforms by T into the boundary in a counter-clock-wise direction of some other region. It has already been pointed out that the T should be chosen so that the group generated by S and T shall be transitive if we expect to construct the whole map by means of S and T. Now if S and T are displacements and if the group generated by them is transitive on all twelve letters, the surfaces on which the maps may be drawn will be one-sided or two-sided according as the group generated by S and T is of order 120, or order 60. Γ'_{120} is the only one of the three groups which contains a transitive G_{60}, so it alone can give rise to maps on a two-sided surface.

The above considerations lead to the following slightly more general statement:

A necessary condition that there exist a map of n k-sided regions on a two-sided surface admitting a group Γ_{2kn} of transformations into itself is that Γ_{2kn} contain a transitive subgroup of index 2.

The connectivity R_1 of the various surfaces is found immediately by the generalized Euler formula $a_0 - a_1 + a_2 = 2 - (R_1 - 1)$, where a_0, a_1, a_2, are the numbers of vertices, edges, and countries in the map. The connectivity of the surface on which the Cremona map lies is 6; Maps III, V, and IX lie on one-sided surfaces of connectivity 9; Maps IV and VIII lie on one-sided surfaces of connectivity 15; Map VI is on a two-sided surface of connectivity 1 and genus p (satisfying the relation $2p + 1 = R_1$) equal to 0; Map VII is on a two-sided surface of connectivity 9 and genus 4. This last surface is the sphere with 4 handles described by Dehn and Heegaard.* The maps

* *Encykl. der Math. Wiss.*, III A. B. 3 (1907), pp. 153-220.

with 4, 6, and 10 countries at a vertex are spheres with 2 ordinary handles and one *one-sided cap* (Kreuzhaube), 3 ordinary handles and 1 *one-sided* handle, and 6 ordinary handles and 1 *one-sided* handle, respectively.

Due to their connection with algebraic curves maps on two-sided surfaces are of particular interest. For example it is known that there exists an algebraic curve possessing a group $G_{5!}$ and having genus 4. As a result of our restriction to maps of 12 five-sided regions we have obtained a small number of maps, and we have obtained no map on a two-sided surface whose group is $G_{5!}$. Knowing that such a map exists and inquiring into possible methods of constructing it, we are led to the theory of covering surfaces.

Any two-dimensional manifold except the sphere may be covered in a (2—1) manner by a two-sided manifold. The construction of the covering surface M_2 may be carried out as follows: Let us consider a surface M_1 which we will take to be one-sided. M_1 may, by a proper choice of a fundamental set of circuits,* be represented on a polygon P_1 of an even number of sides whose vertices correspond to a single point of M_1, and which has one or two pairs of sides with members having like senses on the boundary P_1 according as P_1 does not or does have a number of sides equal to a multiple of four. Now consider a polygon P_2 exactly like P_1, lying over P_1. We may construct a third polygon P_3 by combining P_1 and P_2. We first displace P_2 in its own plane until a side c of P_2 coincides with the side c' of P_1, where cc' form a pair of corresponding sides having the same sense on the boundary of P_1; then we rotate the plane of P_2 through an angle of 180° about the side c, thus leaving fixed every point of the line through c. We let the side c' of P_2 be the one which corresponds to c of P_1 since both correspond to the same line on M_1. If there is a second pair dd' on P_1 which have the same sense we let d of P_2 correspond to d' of P_1 and d' of P_2 correspond to d of P_1. We leave the rest of the correspondences between sides of P_1 and P_2 as they were. The result is that P_3 has an even number of sides which are paired so that the members of each pair are oppositely sensed on the boundary of P_3. The manifold M_2 corresponding to P_3 is therefore two-sided. We will omit the details of showing that the vertices of P_3 constitute two conjugate sets,† and that M_2 thus corresponds to M_1 in a (2—1) manner throughout.

A map of 12 five-sided regions on M_1 gives rise to a map of 24 five-sided regions on M_2. The map on M_2 is regular at its vertices for we may assume

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* H. R. Brahana, *loc. cit.*

† *I. e.* break up into the sets which are imaged on M_2 by two distinct points. For this use of the term conjugate set see Brahana, *locum citum.*

that the point O of M_1 to which the vertices of P_1 correspond was distinct from the vertices of the map. The group of displacements of the map on M_2 is the same as the total group of the map on M_1 as may be seen if we distinguish between the sides of the one-sided surface M_1 in a manner analogous to that used to distinguish between the sheets of a Riemann surface. In the case of Riemann surfaces certain more or less arbitrary lines called branch lines are drawn and crossing a branch-line means going from one sheet to another. In the case of the one-sided surface M_1 we may let the circuits C and D corresponding to edges cc' and dd' of P_1 play a role analogous to that of the branch lines above; crossing C or D means going from one side of M_1 to the other and that is the only way to get from one side to the other. Displacements on the surface M_1 which carry a point across C or D give rise on P_3 to displacements of the corresponding points from P_1 to P_2 or from P_2 to P_1. The displacements on M_1 constitute the total group of M_1; the group of displacements of the map on M_2 is the same as the total group of the map on M_1.

We may construct the following maps. The (2—1) covering surface of the surface containing the Cremona map is a two-sided surface of genus 4 on which we have a map of 24 five-sided regions admitting group $G_{5!}$ of displacements. Maps III, V, IX give rise to maps of 24 five-sided regions on two-sided surfaces of genus 9, having groups $G_{5!}$, G_{120}, G'_{120}, respectively for groups of displacements. Maps IV and VIII give rise to maps on surfaces of genus 13, having groups G_{120}, G'_{120}, respectively for groups of displacements.

The interesting question arises as to whether the groups obtained for these maps are not merely topologic but also analytic as is the case with the dodecahedron map VI. We know that collineation groups abstractly isomorphic with $G_{5!}$ and G_{120} exist in various spaces which have invariant algebraic curves unlimited in number and genus. On such an invariant curve, or better on the allied Riemann surface, a division into 120 fundamental regions is possible. We ask whether this division can be so made that the fundamental regions can be grouped 10 at a time into 12 connected regions (or 5 at a time into 24 connected regions in the case of the doubly covered maps) which constitute one of the maps already found. Since the genus of the invariant curve is unlimited and the genera of the maps limited to 0, 4, 9, 13 this is not always possible. The maps II and III when doubled give rise to maps of 24 five-sided regions with respectively 30 and 20 corners, and genera 4 and 9. These undoubtedly arise from the algebraic curves which in space are determined by the equations

$$\text{(II)} \quad \Sigma y_i = 0, \quad \Sigma y_i^2 = 0, \quad \Sigma y_i^3 = 0;$$
$$\text{(III)} \quad \Sigma y_i = 0, \quad \Sigma y_i^2 = 0, \quad \Sigma y_i^4 = 0; \quad (i = 0, \cdots, 4);$$

and are invariant under the collineation group of the permutations of the y's. Thus in (II) the 30 corners must each be invariant under a cyclic G_4 and these must be the 30 points whose coordinates are a permutation of 0, 1, i, -1, $-i$—points which evidently satisfy (II). In (III) the 20 corners must each be invariant under a cyclic G_3 and a permutable G_2 and these must be the points whose coordinates are a permutation of 0, 0, 1, ω, ω^2 ($\omega = e^{2\pi i/3}$)—points which are on the curve (III).

Of curves with a G_{120} we note the one of genus 9 with the equation

$$y = \sqrt{(ax)^{20}},$$

where $(ax)^2 = 0$ represents in the complex plane of x the projection of the 20 vertices of the dodecahedron. This curve is hyperelliptic with an ikosahedral g_{60} and a permutable element U which is $y' = -y$, $x' = x$. The 20 branch points may in this case be the corners of the doubled maps V and IX, one or both maps arising according to the manner in which the 120 fundamental regions are grouped.

There seems to be no algebraic curve which satisfies the group requirements for the production of the map VII of genus 4 nor for the doubled maps IV and VIII of genus 13.

Fundamental regions for the Cremona map II have been exhibited by H. E. Slaught (*Amer. Jour. Math.*, Vol. 22 (1900), pp. 343-80) and the map II itself arises by grouping sets of 10 of his fundamental regions as may be verified by an inspection of his Figs. X, XI.

URBANA, ILLINOIS,
June 1, 1925.

On Surfaces and Curves which are Invariant under Involutory Cremona Transformations.

By Arnold Emch.

I. n-Space.

1. *Introduction.* Let $(x) = (x_1, x_2, x_3, \cdots, x_{n+1})$ and $(x') = (x'_1, x'_2, x'_3, \cdots, x'_{n+1})$ define two points in two projective n-spaces Σ and Σ' which are superposed, i. e., which have the same reference polyhedron $A_1\ (1, 0, 0, \cdots, 0),\ \cdots, A_{n+1}\ (0, 0, 0, \cdots, 1)$.

Between Σ and Σ' we assume an involutory Cremona transformation such that when

$$(1) \qquad \rho x'_i = \phi_i(x_1, x_2, x_3, \cdots, x_{n+1}),\ i = 1, 2, \cdots, n+1$$

in which the ϕ_i's are homogeneous polynomials of degree m in the x's, there is, by definition of the involutory property, also

$$(2) \qquad \sigma x_i = \phi_i(x'_1, x'_2, x'_3, \cdots, x'_{n+1}),\quad i = 1, 2, \cdots, n+1.$$

To a point P in Σ corresponds, in general, one and only one point P' in Σ', and conversely. If this same point P' is chosen as a point of Σ, then to it corresponds the originally chosen point P in Σ'. There are exceptional points, lines, and surfaces, for which the transformation ceases to be definitely one-to-one. These form the *fundamental-points, -lines,* and *-surfaces,* etc., and depend on the nature of the functions ϕ_i.

Every couple of correspoding points, P, P' determines a line l in n-space, so that the totality of such couples forms an n-dimensional line-manifold. In this manifold are contained in particular those sub-manifolds which are formed by the joins of corresponding points on invariant surfaces and curves of the involution.

As in three-spaces, we may form the Plückerian homogenous coordinates of a line in n-space. Such a line l is determined by two points

$$(y) = (y_1, y_2, y_3, \cdots, y_{n+1})$$
$$(z) = (z_1, z_2, z_3, \cdots, z_{n+1}).$$

From the matrix of the y's and z's $\dfrac{n(n+1)}{2}$ expressions

$$\rho p_{ik} = y_i z_k - y_k z_i$$

may be formed. These may be defined as the homogeneous coordinates of l. Among the p_{ik}'s exist $\frac{(n+1)n(n-1)(n-2)}{24}$ identities of the form

$$p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0.$$

By means of these identities, all the p_{ik}'s may be expressed in terms of $2n-1$ definitely chosen ones.

We may, in particular construct the coordinates of the join of two corresponding points of the involution:

$$\rho p_{ik} = x_i\phi_k - x_k\phi_i. \tag{3}$$

In case of an involution there is

$$\sigma x_i'' = \phi_i\left[\frac{\phi_i(x)}{\rho}\right] = kJ.x_i,$$

where J denotes the Jacobian of the ϕ's, so that in the involution the coordinates (3), except as to the same reproducing factor kJ, and consequently the hypersurface, represented by an hypercomplex

$$K_w(p_{12}, p_{13}, \cdots, p_{ik}) = 0, \tag{4}$$

remains invariant. If this complex is not reducible or not of a special type, the invariant hypersurface is of order $w(m+1)$.

It is clear that if the involution has a fixed locus this (if of dimension $n-1$) will be a factor of the p_{ik}'s.

The purpose of this paper is to investigate, in its general aspect, the properties of these transformations and their bearing upon invariant surfaces and curves, and to apply the theory to some specific cases. The converse problem, to determine those involutions which leave given complexes, surfaces, and curves invariant, although of considerable importance and solved for some particular cases, will not be considered here.

2. *The Two Types of Invariant Hypersurfaces.*

In the involution of $x'_i = \phi_i(x)$ the $p_{ik} = x_i\phi_k - x_k\phi_i$ are members of a linear invariant system.

Under the same conditions this is also true of the functions

$$\sigma g_{ik} = x_i \phi_k + x_k \phi_i \tag{5}$$

including the functions $\sigma g_{ii} = 2x_i \phi_i$, obtained for $k = i$.

The reproducing factors of the p_{ik}'s and g_{ik}'s are of opposite sign. Hence also

$$L_w(g_{11}, g_{13}, \cdots, g_{ik}, \cdots, g_{ii}, \cdots, g_{n+1,n+1}) = 0, \tag{6}$$

in which the g_{ik}'s from (5) are substituted represents an invariant hypersurface.

The problem is to ascertain to what extent the two families of surfaces (4) and (6) differ from each other. From (3) and (5) the identity

$$p_{ik}p_{jl} = g_{il}g_{jk} - g_{ij}g_{kl} \tag{7}$$

follows, containing for $j = i$, $l = k$, the case

$$p^2{}_{ik} = g^2{}_{ik} - g_{ii}g_{kk}. \tag{8}$$

From this follows that any equation of even degree in the p_{ik}'s may be rationally expressed in terms of the g_{ik}'s. Hence an equation of odd degree in the p_{ik}'s is not expressible in the g_{ik}'s, but its square is. Thus when $K_{2\mu+1}(x) = 0$ represents a hypersurface of the first type, $[K_{2\mu+1}(x)]^2 = L_\nu(x)$, where $\nu = 2(2\mu + 1)$ belongs to the second type. Hence

THEOREM 1. *In every involutory Cremona transformation in n-space there are two types of invariant surfaces (curves in case of 2-space representable by the forms*

$$K(p_{12}, \cdots, p_{ik}, \cdots) = 0,$$
$$L(g_{11}, \cdots, g_{ik}, \cdots, g_{ii}, \cdots) = 0.$$

When K is of even degree in the p_{ik}'s, then it may be classed with the form L. When K is of odd degree, then its square K^2 only belongs on the class L. Accordingly invariant hypersurfaces belong to two classes: class 1, hypersurfaces which belong to L but are not squares of K; class 2, hypersurfaces whose squares only belong to L.

Suppose that $N + 1$ is the maximum number of linearly independent g_{ik}'s, $g_1, g_2, \cdots, g_{N+1}$, i. e., of invariant polynomials of degree $n + 1$ with

the same reproducing factor as in (5), which we consider as projective coordinates of a point in N-space. Then to every couple $P(x)$, $P'(x')$, or their join l, of the transformation corresponds in N-space a definite point Q of an n-spread Γ. A hypersurface $F(g_1, g_2, \cdots, g_{N+1}) = 0$ in N-space cuts Γ in an $n-1$ spread S to whose points correspond in n-space couples of corresponding points on the invariant hypersurface $F(x_i\phi_k + x_k\phi_i, \cdots, g_i, \cdots) = 0$, determined by $F(g_1, g_2, \cdots, g_{N+1}) = 0$. A hypersurface $L(g_1, g_2, \cdots, g_{N+1}) = 0$ which results as the square of a hypersurface $K(p_{12}, \cdots) = 0$ of odd degree touches (has a contact of the first order) along an $n-1$ spread S, to which corresponds in n-space an invariant hypersurface counted doubly.

Thus *the distinctive differences between the two types of invariant hypersurfaces reveal themselves in a contact-problem.*

This important aspect of invariant forms in involutory Cremona transformations will be considered more fully in some specific cases.

II. EXAMPLES IN THE PLANE.

A. *The Quadratic Transformation.*

3. *The Curves C_n and K_m.* The quadratic involutory transformation [(1), 1] may be assumed in the simple form

$$\rho x_1' = x_2x_3, \quad \rho x_2' = x_3x_1, \quad \rho x_3' = x_1x_2. \tag{9}$$

For the sake of simplicity, in place of the notation $\rho p_{ik} = x_i\phi_k - x_k\phi_i$ we write $u_l = p_{ik}$, likewise $v_l = g_{ik}$, $i, k, l = 1, 2, 3$. The invariant cubics of the two types are

$$a_1x_1(x_2^2 - x_3^2) + a_2x_2(x_3^2 - x_1^2) + a_3x_3(x_1^2 - x_2^2) = 0 \tag{10}$$

and

$$a_1x_1(x_2^2 + x_3^2) + a_2x_2(x_3^2 + x_1^2) + a_3x_3(x_1^2 + x_2^2) + a_4x_1x_2x_3 = 0, \tag{11}$$

of which (10) is of particular importance in the study of invariant curves of higher order. In place of the hypercomplex (4), K_m we have the curve K_m of class m which gives rise to an invariant curve C_n, in which n and m are in a certain dependence as will presently be shown. In three recent papers *

* *The Tohoku Mathematical Journal*, Vol. 21, pp. 210-226 (1922); Vol. 24, pp. 68-87 (1924); Vol. 25, pp. 63-76 (1925).

the author has based the investigation of invariant C_n's on the fact that joins (u) of couples of corresponding points on C_n envelope a certain class-curve K_m, and that conversely every invariant C_n may be generated from such a class-curve, in case of involutions of class (or rank) 1. A discussion and classification of all invariant curves up to and including order six has been fully worked out.

In what follows particular attention will be given to the relation between the p_{ik} and q_{ik} manifolds and the p_i and (x) spaces; in the case of the plane betwen the (u)- and (x)-planes and the (v)-space.

We shall denote by $A_1(1, 0, 0)$, $A_2(0, 1, 0)$, $A_3(0, 0, 1)$ the fundamental-points, by $B(1, 1, 1)$ (unit-point), $B_1(-1, 1, 1)$, $B_2(1, -1, 1)$, $B_3(1, 1, -1)$ the invariant points of the transformation. To A_1, A_2, A_3 correspond respectively the lines A_2A_3, A_3A_1, A_1A_2. Conversely to every point of A_iA_k corresponds the opposite point A_j. The lines BB_1, BB_2, BB_3, B_1B_2, B_2B_3, B_3B_1 are invariant in such a manner that two corresponding points P and P' on B_iB_k form a harmonic quadruple with B_i and B_k.

By (9), a C_n is, in general, transformed into a C_{2n}. Hence as a necessary condition for an invariant C_n, the C_n must have multiple points of order a_1, a_2, a_3 at A_1, A_2, A_3 respectively so that

$$a_1 + a_2 + a_3 = n. \tag{12}$$

Let moreover C_n have multiplicities of orders β, β_1, β_2, β_3 at B, B_1, B_2, B_3.

The class m is the number of variable corresponding pairs of C_n on a line through S, i. e., $2m$ is the number of variable intersections with C_n of the isologous cubic of S, or

$$2m = 3n - a_1 - a_2 - a_3 - \beta - \beta_1 - \beta_2 - \beta_3. \tag{13}$$

The multiplicity μ_1 of the line B_2B_3 is half the number of intersections of C_n with B_2B_3 outside B_2, B_3, A_1, i. e.,

$$\begin{aligned} 2\mu_1 &= n - \beta_2 - \beta_3 - a_1, \\ 2\mu_2 &= n - \beta_3 - \beta_1 - a_2, \\ 2\mu_3 &= n - \beta_1 - \beta_2 - a_3. \end{aligned} \tag{14}$$

Likewise for the multiplicities ν_i on the lines BB_i there is

$$\begin{aligned} 2\nu_1 &= n - \beta - \beta_1 - a_1, \\ 2\nu_2 &= n - \beta - \beta_2 - a_2, \\ 2\nu_3 &= n - \beta - \beta_3 - a_3. \end{aligned} \tag{15}$$

Since $\overline{BB_1}$, $\overline{BB_2}$, $\overline{BB_3}$ are on B the sum of their multiplicities must be not greater than the class m, or

$$(16)\qquad \begin{aligned} &\nu_1+\nu_2+\nu_3 \gtrless m, \text{ whence also}\\ &\nu_1+\mu_2+\mu_3 \gtrless m,\\ &\cdots\cdots\cdots\\ &\mu_1+\mu_2+\nu_3 \gtrless m. \end{aligned}$$

The associated K-curve is therefore of class m.

Conversely assume a K_m with B_1B_2, B_2B_3, B_3B_1, BB_1, BB_2, BB_3 as multiple tangents of multiplicity μ_3, μ_1, μ_2, ν_1, ν_2, ν_3 respectively. With a K_m is in general associated a C_n of order $n=3m$. This is the case when the lines B_iB_k are not tangents of K_m. If they are tangents of multiplicity as stated above, then they split off with the same multiplicities from C_{3m}, so that the invariant C_n proper has the order

$$(17)\qquad n=3m-\Sigma\mu-\Sigma\nu,$$

so that

$$(18)\qquad \Sigma\mu+\Sigma\nu=3m-n.$$

There also exist the equalities

$$m-\mu_1-\nu_1=a_1,\quad m-\mu_2-\nu_2=a_2,\quad m-\mu_3-\nu_3=a_3,$$

and the inequalities

$$\Sigma\nu\leq m,\quad \mu_1+\mu_2+\nu_3\leq m,\quad \mu_2+\mu_3+\nu_1\leq m,\quad \mu_3+\mu_1+\nu_2\leq m.$$

From (18) it is seen that when m and n are given we know $\Sigma\mu+\Sigma\nu$, i. e., the sum of the multiplicities of the tangents B_iB_k. Obviously, $3m-n\geq 0$. But there is an upper limit for m when n is given. In fact it can be shown that $m\leq n$. Suppose the contrary, $m>n$, i. e., $m=n+e$, where e is a positive integer. Then from (18)

$$(a)\qquad \Sigma\mu+\Sigma\nu=2n+3e.$$

Summing the inequalities $\mu_1+\mu_2+\nu_3\leq m$, etc., we have

$$(b)\qquad 2\Sigma\mu+\Sigma\nu\leq 3n+3e.$$

Also

(c) $$\Sigma\nu \leq n + e.$$

Now subtract (b) from (a), giving

(d) $$\Sigma\mu \leq n.$$

Add (c) and (d), giving

(e) $$\Sigma\mu + \Sigma\nu \leq 2n + e.$$

(a) and (e) can possibly be consistent only when $e = 0$, i. e., finally

$$m \leq n.$$

This result may be stated by

THEOREM 2. *An invariant C_n cannot be generated from a K_m whose class m is greater than n.*

Equation (18) gives us a means to determine the class of the K_m's from which the C_n may be generated. For example when we have a sextic C_6, $\Sigma\mu + \Sigma\nu = 3m - 6$, $m \leq 6$. We have the possibilities $m = 2, 3, 4, 5, 6$ so that invariant sextics may be generated from class-curves K_2, K_3, K_4, K_5, K_6. On the other hand, for a given C_n, there must be $\Sigma\beta = n$; in the case of a sextic, for example, $\Sigma\alpha = 6$.

4. *The Involutory Quadratic Transformation and the Cayley Cubic Surface.* Consider a point plane (x) and a line plane (u) referred to the same coordinate triangle, so that the line $x_1 + x_2 + x_3 = 0$ is the unit line of the (u)-plane and $u_1 + u_2 + u_3 = 0$ is the unit point of the (x)-plane.

(x_1, x_2, x_3) and (x_2x_3, x_3x_1, x_1x_2) are two corresponding points of the quadratic transformation. The line coordinates of the line joining these two points are

(19) $$\rho u_1 = x_1(x_2^2 - x_3^2), \quad \rho u_2 = x_2(x_3^2 - x_1^2),$$
$$\rho u_3 = x_3(x_1^2 - x_2^2).$$

There exists in general a (1, 1) correspondence between couples of corresponding points of the transformation and the lines of the (u)-plane. To a class-curve $K_m(u) = 0$ corresponds in general an invariant $C_n(x) = 0$ whose order is $n = 3m$. How this number is reduced for particular positions of K_m with respect to the coordinate triangle was shown in the preceding section.

Now it is noticed that by the quadratic transformation the signs of the u's in (19) are changed, so that for this reason the curve $\Sigma a_i x_j (x_i^2 - x_k^2) + a_4 x_1 x_2 x_3 = 0$ is not invariant. The line (u_1, u_2, u_3) itself remains unchanged. On the other hand the quantities

$$(20) \qquad \begin{aligned} \rho v_1 &= x_1(x_2^2 + x_3^2), & \rho v_2 &= x_2(x_3^2 + x_1^2), \\ \rho v_3 &= x_3(x_1^2 + x_2^2), & \rho v_4 &= 2x_1x_2x_3 \end{aligned}$$

are not changed by the transformation. From this it is seen that with every couple of corresponding points of the transformation there is uniquely associated a quadruple (v_1, v_2, v_3, v_4) which defines a point in a quaternary v-space. There are ∞^2 couples of corresponding points in the (x)-plane so that the corresponding points (v) must form a surface, whose quation is easily found to be

$$(21) \qquad \Gamma_3 = 2v_1v_2v_3 - (v_1^2 + v_2^2 + v_3^2)v_4 + v_4^3 = 0.$$

This is a cubic with four nodes, and hence a Cayley * cubic whose nodes correspond to the invariant points B, B_1, B_2, B_3 of the quadratic transformation. In fact to $B(1, 1, 1)$, $B_1(-1, 1, 1)$, $B_2(1, -1, 1)$, $B_3(1, 1, -1)$ in the (x)-plane correspond in (v)-space the points $B^*(1, 1, 1, 1)$, $B_1^*(-1, 1, 1, -1)$, $B_2^*(1, -1, 1, -1)$, $B_3^*(1, 1, -1, -1)$ and we can immediately verify that these are conical nodes of (21). Denoting the vertices of the coordinate tetrahedron in (v) by $V_1V_2V_3V_4$, it is found that to the line $x_3 = 0$ corresponds in (v)

$$\rho v_1 = x_1x_2x_2, \qquad \rho v_2 = x_1x_2x_1, \qquad \rho v_3 = 0, \qquad \rho v_4 = 0,$$

i. e., the line V_1V_2; likewise to $x_1 = 0$ and $x_2 = 0$ correspond in (v) the lines V_2V_3 and V_3V_1 respectively. The points $A_1(1, 0, 0)$, $A_2(0, 1, 0)$ $A_3(0, 0, 1)$ are the fundamental points of the transformation. To them correspond in (v) the lines V_2V_3, V_3V_1, V_1V_2. To the invariant lines BB_1, BB_2, BB_3, B_1B_2, B_2B_3, B_3B_1 correspond in (v) in the same order the edges of the nodal tetrahedron $B^*B_1^*B_2^*B_3^*$. Denoting the intersections of the edges V_iV_k of the coordinate tetrahedron with the unit plane of (v), $(v_1 + v_2 + v_3 + v_4 = 0)$, by E_{ik}, these form the six vertices of a quadrilateral whose dia-

* It seems proper to call this surface the Cayley cubic, since Cayley was the first who investigated this surface. *Journal de Mathématiques pures et appliquées*, Vol. 9, pp. 285-293 (1844).

gonal points are precisely the points B_1^*, B_2^*, B_3^* while B^* is the unit-point of (v). Each edge $B_i^*B_k^*$, $i, k = 0, 1, 2, 3$, of the nodal tetrahedron cuts one of the sides V_1V_2, V_2V_3, V_3V_1. To a point $(v_1, v_2, 0, 0)$ of V_1V_2 corresponds in (x) a figure determined by (20), with $v_3 = v_4 = 0$. These are satisfied when $x_3 = 0$, $x_2 = \rho v_1$, $x_1 = \rho v_2$, i. e., to such a point corresponds a point $(v_2, v_1, 0)$ on A_1A_2, but also the point A_3. Thus to a point on one of the edges V_iV_k correspond a point on A_iA_k and the point A_j opposite A_iA_k. To a point on one of the edges $B_i^*B_k^*$ corresponds on B_iB_k a couple of harmonic points with respect to B_i and B_k. To every other point of the Cayley cubic corresponds in (x) a couple of corresponding points of the quadratic transformation. The lines of the cubic are the six edges of the nodal tetrahedron and the sides of the triangle $V_1V_2V_3$.

To every curve on the Γ_3 corresponds an invariant curve in (x). Such a curve may be cut out by a surface $F(v) = 0$, so that the equation of the invariant C_n in (x) is

$$F\{x_1(x_2^2 + x_3^2),\ x_2(x_3^2 + x_1^2),\ x_3(x_1^2 + x_2^2),\ 2x_1x_2x_3\} = 0,$$

which is of course reducible according as the $F(v)$ passes or does not pass through the edges of the nodal tetrahedron or the sides of $V_1V_2V_3$ with definite multiplicities.

5. *Relation between the (u)-plane and the Cayley Cubic.* From the transformations (19) and (20) follows immediately

$$\begin{aligned}(22)\qquad &\rho u_1^2 = v_1^2 - v_4^2, \quad \rho u_2^2 = v_2^2 - v_4^2, \quad \rho u_3^2 = v_3^2 - v_4^2,\\ &\rho u_1u_2 = -v_1v_2 + v_2v_3, \quad \rho u_2u_3 = -v_2v_3 + v_1v_4,\\ &\rho u_3u_1 = -v_3v_1 + v_2v_4.\end{aligned}$$

It is verified that to the edge $v_1 = 0$, $v_4 = 0$ correspond the lines $\rho u_1 = 0$, $\rho u_2 = v_2$, $\rho u_3 = v_3$ through A_1, i. e., in the quadratic correspondence A_1 and the opposite side A_2A_3. Similar relations hold with respect to the remaining vertices and edges in the $(v_4 = 0)$-plane. The equations of the faces of the nodal tetrahedron are

$$\begin{aligned}B^*B_1^*B_2^* \quad & v_1 + v_2 - v_3 - v_4 = 0\\ B^*B_2^*B_3^* \quad & -v_1 + v_2 + v_3 - v_4 = 0\\ B^*B_3^*B_1^* \quad & v_1 - v_2 + v_3 - v_4 = 0\\ B_1^*B_2^*B_3^* \quad & v_1 + v_2 + v_3 + v_4 = 0.\end{aligned}$$

For the edge B_1B_2, for example, we have $v_2 = -v_1$, $v_4 = -v_3$, to which corresponds in the u-plane $\rho u_1^2 = v_1^2 - v_3^2$, $\rho u_2^2 = v_1^2 - v_3^2$, $\rho u_3^2 = 0$, or $\sigma u_1 = \pm\sqrt{v_1^2 - v_3^2}$, $\sigma u_2 = \pm\sqrt{v_1^2 - v_3^2}$, $\sigma u_3 = 0$. But as $\rho u_1 u_2 = v_1^2 - v_3^2$, we must choose for both square roots the same sign, so that to B_1B_2 corresponds the line $u_1 = 1$, $u_2 = 1$, $u_3 = 0$, or B_1B_2. In general, to $B_i{}^*B_k{}^*$ corresponds in the u-plane B_iB_k. The results may be summed up in

THEOREM 3. *There exists a one-to-one correspondence between the couples of corresponding points of an involutory quadratic Cremona transformation and the points of a Cayley cubic. To the lines of the nodal tetrahedron correspond the invariant lines of the quadratic transformation, and conversely. Every line of the nodal tetrahedron cuts one of the remaining three lines of the cubic. To a point on one of these three lines (V_1V_2, V_2V_3, V_3V_1) corresponds a point on the respective lines A_1A_2, A_2A_3, A_3A_1, and the opposite base-point.*

To every point of the Cayley cubic corresponds a line in the u-plane, and conversely. To an edge of the nodal tetrahedron corresponds the line with the same indices invariant in the quadratic transformation.

To a $K_m(u_1, u_2, u_3)$ corresponds a curve S on Γ_3 cut out by a certain surface (or surfaces) $F(v_1, v_2, v_3, v_4) = 0$. To S corresponds on the other hand an invariant curve C_n in (x). Conversely a surface F cuts the Γ_3 in a curve to which corresponds in (x) an invariant C_n which may be generated from a definite K_m.

It is important to specify the passage from a $K_m(u)$ to the corresponding $F(v)$, and conversely from an $F(v)$ to the associated K_m. From (22) it is apparent that a K_m which is rational in u_1^2, u_2^2, u_3^2, u_1u_2, u_2u_3, u_3u_1 can immediately be expressed as a rational function F_m of v_1, v_2, v_3, v_4 of the same degree, so that the F_m cuts the Γ_3 in a curve which corresponds to the C_n attached to K_m. This will always be the case when m is even. To prove this, let $u_1^{\alpha}u_2^{\beta}u_3^{\gamma}$ be any term of K_m such that $\alpha + \beta + \gamma = m = 2k$. Then we have the possibilities $\alpha = 2\lambda + 1$, $\beta = 2\mu + 1$, $\gamma = 2\nu$, and $u_1^{\alpha}u_2^{\beta}u_3^{\gamma} = (u_1^2)^{\lambda}(u_2^2)^{\mu}(u_2^2)^{\nu}u_1u_2$, and $\alpha = 2\lambda$, $\beta = 2\mu$, $\gamma = 2\nu$ with $u_1^{\alpha}u_2^{\beta}u_3^{\gamma} = (u_1^2)^{\lambda}(u_2^2)^{\mu}(u_2^1)^{\nu}u_1u_2$, and $\alpha = 2\lambda$, $\beta = 2\mu$, $\gamma = 2\nu$ with $u_1^{\alpha}u_2^{\beta}u_3^{\gamma} = (u_2^1)^{\lambda}(u_2^2)^{\mu}(u_3^2)^{\nu}$, which proves the proposition. But when $\alpha + \beta + \gamma = m = 2k + 1$, we have either $\alpha = 2\lambda + 1$, $\beta = 2\mu + 1$, $\gamma = 2\nu + 1$ with $u_1^{\alpha}u_2^{\beta}u_3^{\gamma} = (u_1^2)^{\lambda}(u_2^2)^{\mu}(u_3^2)^{\nu}u_1u_2u_3$, or $\alpha = 2\lambda$, $\beta = 2\mu$, $\gamma = 2\nu + 1$ with $u_1^{\alpha}u_2^{\beta}u_3^{\gamma}$

$= (u_1^2)^\lambda (u_2^2)^\mu (u_3^2)^\nu u_3$. Hence K_m cannot be expressed rationally in terms of the v's in this case. This however becomes possible for the square $(K_m)^2$, i. e., the K_m counted twice. The corresponding $F_{2m}(v) = 0$ becomes a tac-surface of the Γ_3, so that the intersection of F_{2m} on Γ_3 is an S_{3m} space curve counted twice. This is, of course, merely a verification of the general theory. Thus

THEOREM 4. *When $K_m(u)$ is of even class then to the attached invariant C_n corresponds on Γ_3 an S_n cut out by an $F_m(v)$. When $K_m(u)$ is of odd class, then to the attached invariant C_n corresponds on Γ_3 the curve of a tac-surface F_{2m}. Eventual reductions in the number n are caused by the sides of the invariant quadrangle which may be single or multiple tangents of K_m, or accordingly by the edges of the nodal tetrahedron of Γ_3 which may be single or multiple lines of F_m or F_{2m}.*

If an F_m cuts a line V_iV_k in m points outside of V_i and V_k, then the corresponding $C_n(x)$ has a multiple point of order m at A_j, $(i, j, k = 1, 2, 3)$.

The principle involved in the application of the relation between C_n and K_m curves and the Cayley cubic Γ_3 for the study of invariant curves may be extended to involutory plane Cremona transformations of any order, as is of course obvious from the general theory in I.

B. *The Geiser Transformation.**

6. *Definitions and General Properties.* This involutory Cremona transformation may be defined as the correspondence between the points of a couple (P, P') obtained by the variable intersections of any two cubics through seven fixed points $A_1, \cdots, A_7$. These are the fundamental-points of the transformation. To a fundamental-point A_i corresponds as a principal-curve the nodal cubic with its node at A_i and passing singly through the six remaining fundamental-points. The Jacobian of the net is obviously a pointwise invariant curve and is a sextic of genus 3 whose seven nodes coincide with the nodes of the principal-cubics. It is known as the Aronhold curve uniquely attached to the seven fundamental-points. I shall denote it by I_6.

The transformation is of order 8 and class 1, i. e., on every line there is

* Geiser, "Ueber zwei geometrische Probleme," *Journal für reine und angewandte Mathematik*, Vol. 67, pp. 78-89 (1867).

in general one couple (P, P') of corresponding points. The homoloidal net consists of the octics with triple points at each of the fundamental-points. The Jacobian of this net is composed of all principal-cubics. If $\phi_1 = 0$, $\phi_2 = 0$, $\phi_3 = 0$ are three linearly independent octics of the net, the analytic form of the transformation is $\rho x_i' = \phi_i$, $i = 1, 2, 3$. To a line $a_1x_1 + a_2x_2 + a_3x_3 = 0$ corresponds to the octic $a_1\phi_1 + a_2\phi_2 + a_3\phi_3 = 0$ which cuts the line in eight points, of which six are on I_6 while the remaining two form the couple (P, P'). $\rho u_j \doteq x_i\phi_k - x_k\phi_i$ are the coordinates of the line joining the couple $[P(x), P'(x')]$. On every line of the pencil $u_1y_1 + u_2y_2 + u_3y_3 = 0$ through (y) there is a couple (P, P'). The locus of these couples is a cubic C_3 whose equation is a factor of the nonic

$$\sum y_j(x_i\phi_k - x_k\phi_i) = c.I_6.C_3 = 0, \tag{23}$$

in which the other factor I_6 gives the Aronhold curve. C_3 is invariant, passes through (y) and is called the "isologue" of (y).

7. *Invariant Curves.* As in the quadratic transformation all invariant C_n's, eventually counted twice, may be generated from K_m class-curves. If in $K_m(u) = 0$ we substitute for the u's the expressions of degree nine in the x's we obtain a curve of order $9m$ from which splits off the Aronhold curve I_6 counted m times, so that in general the order of the non-pointwise invariant C_n is $n = 9m - 6m = 3m$. Denoting by $C_3^{(1)}$, $C_3^{(2)}$, $C_3^{(3)}$ the isologues of the points $u_1 = 0$, $u_2 = 0$, $u_3 = 0$, the equation of the invariant C_{3m} becomes

$$K_m(C_3^{(1)}, C_3^{(2)}, C_3^{(3)}) = 0. \tag{24}$$

Consider for example the class of invariant sextics with double points at the A_i's. They are generated from the class of conics $K_2 = 0$ and form therefore a linear ∞^5 manifold. But as there are ∞^6 sextics with double points at the A_i's, there exist non-invariant sextics with double points at the A_i's.

A cubic isologue C_3 attached to a point (y) cuts the I_6 outside of the A_i's in four points, representing four couples of coincident corresponding points on the C_3 and on the I_6. Hence the tangents to the C_3 at these points pass through (y). From this follows that the joins of couples of coincident corresponding points on the I_6 envelope a class-quartic K_4, or that the I_6 is generated from a K_4. A line l through a base-point A_i cuts the corresponding principal-curve in a point B_i outside of A_i. To B_i corresponds A_i.

Hence when B_i approaches A_i on the principal-curve, l approaches the tangent to a branch of the I_6 through A_i and becomes a tangent to K_4. Hence

THEOREM 5. *In the Geiser transformation the Aronhold curve is generated from a quartic of class 4 which is determined by the 7 pairs of tangents at the double points of the Aronhold curve.*

From this follows that

$$K_4(C_3^{(1)}, C_3^{(2)}, C_3^{(3)}) = (I_6)^2, \tag{25}$$

i. e., only the square of I_6 can be rationally expressed by the C_3's. The non-invariant sextic has the form

$$C_6 + \lambda I_6 = 0 \tag{26}$$

where $C_6 = 0$ is an invariant sextic and depends on 5 effective constants, as has been already stated. As C_6 and I_6 are invariant, the transformed of (23) has necessarily the form

$$C_6 - \lambda I_6 = 0.$$

A similar result is easily found with respect to the curves of order 9 with triple points at the A_i's. Such a curve contains 12 effective constants. The invariant C_9 has 9 effective constants, being generated from a K_3. The invariant C_3 depends on 2 constants. Thus, denoting by C_9^* a non-invariant curve of order 9, the C_9 may be put in the form

$$C_9^* = C_9 + \lambda I_6 . C_3 = 0. \tag{27}$$

The transformed is

$$C_9^{*\prime} = C_9 - \lambda I_6 . C_3 = 0.$$

As to the $\sigma q_j = \sigma q_{ik} = x_i \phi_k + x_k \phi_i$, there are six expressions of this kind which belong to the 10 linearly independent C_9's. If we put

$$\sigma q_i = C_9^{(i)}, \quad i = 1, \cdots, 10,$$

the couples (P, P') of the Geiser transformation may be mapped upon a surface Γ of order 9 in a projective nine-space. The geometry on this surface

becomes thus identical with the geometry of invariant figures in the Geiser plane. For the sake of brevity, the discussion of this mapping process may be here omitted.

III. EXAMPLES IN SPACE.

A. *General Involution.*

8. *Order of Complex of an Involution.* For $n = 3$ there are six homogeneous line-coordinates $\rho p_{ik} = x_i\phi_k - x_k\phi_i$ satisfying the single identical equation $p_{12}p_{34} + p_{13}p_{42} + p_{14}p_{23} = 0$. The elimination of x_1, x_2, x_3, x_4 among these equations leads to the equation of the complex $K(p_{12}, \cdots, p_{34}) = 0$, which leads to

THEOREM 6. *The complex of lines joining corresponding points of an involution of order n in space is in general of order n.*

This may be verified as follows. M. Panelli * has shown that the complex associated with a Cremona transformation in which to a plane in Σ corresponds a surface of order n in Σ' and to a plane in Σ' a surface of order n' in Σ is of order $n + n'$. There are $n + n'$ points P such that the corresponding point P' is collinear with P and a fixed point O. But in an involution all points P' are among the $n + n'$ points and moreover $n = n'$, so that the number of joins of P and P' through O reduces to n.

9. *The System* $L(q_{ik})$. There are in general 9 distinct $\rho q_{ik} = x_i\phi_k + x_k\phi_i$ expressions which are invariant, except as to a definite reproducing factor. They are linearly independent. But they do not form a complete set of linearly independent functions of the same order, in general. An instance of this sort in the case of the plane was met in the Geiser transformation. Thus when $q_1, q_2, \cdots, q_N$ form a complete series of linearly independent invariant functions of order $n + 1$ with the same reproducing factor,

$$\sum_{i=1}^{N} a_i q_i = 0$$

represents the most general invariant surface of order $n + 1$. That there are invariant surfaces of lower order possible may be shown by many examples,

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* "Sui complessi associati ad ogni trasformazione birazionale dello spazio," *Giornale di Matematiche*, Vol. 28, pp. 245-286 (1890).

as for instance in the next section. It is not necessary to state the results obtained for general spaces for ordinary space.

B. *A Special Cubic Involutory Transformation.*

10. *Definition and General Properties.* As another application of the general theory consider the well-known transformation

$$\begin{aligned} \rho x_1' &= a_1 x_2 x_3 x_4, & \rho x_2' &= a_2 x_1 x_3 x_4, \\ \rho x_3' &= a_3 x_1 x_2 x_4, & \rho x_4' &= a_4 x_1 x_2 x_3. \end{aligned} \tag{28}$$

To the points of the face of the coordinate tetrahedron, say $x_4 = 0$, corresponds the opposite vertex $A_4(0, 0, 0, 1)$, so that $A_1A_2A_3A_4$ are the fundamental-points of the transformation. To a point on an edge, say on A_3A_4 $(\epsilon_1, \epsilon_2, x_3, x_4; \epsilon_1 \to 0, \epsilon_2 \to 0)$, corresponds, when neglecting infinitesimals of the second order, the point $(a_1\epsilon_2 x_3 x_4, a_2\epsilon_1 x_3 x_4, 0, 0)$, i. e., every point on A_1A_2 depending on the ratio $a_1\epsilon_2/a_2\epsilon_1$. To the plane

$$b_1x_1' + b_2x_2' + b_3x_3' + b_4x_4' = 0 \tag{29}$$

corresponds the Cayley cubic

$$a_1b_1x_2x_3x_4 + a_2b_2x_1x_3x_4 + a_3b_3x_1x_2x_4 + a_4b_4x_1x_2x_3 = 0. \tag{30}$$

The Jacobian of the ϕ_i's $(\rho x_i' = \phi_i)$ is $J = -3x_1^2x_2^2x_3^2x_4^2$, and its vanishing is geometrically identical with the four coordinate faces as principal-surfaces counted doubly. The sextic principal-curve of the more general cubic transformation here degenerates into the six edges of the coordinate tetrahedron.

For the pointwise invariant figure we have from (25) the conditions

$$a_1x_4^2 - a_4x_1^2 = 0, \quad a_2x_4^2 - a_4x_2^2 = 0, \quad a_3x_4^2 - a_4x_3^2 = 0,$$

which are simultaneously satisfied by the eight points

$$\begin{aligned} B_1 &\ (-\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}) \\ B_2 &\ (\sqrt{a_1}, -\sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}) \\ B_3 &\ (\sqrt{a_1}, \sqrt{a_2}, -\sqrt{a_3}, \sqrt{a_4}) \\ B_4 &\ (\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}) \\ C_1 &\ (\sqrt{a_1}, -\sqrt{a_2}, -\sqrt{a_3}, \sqrt{a_4}) \\ C_2 &\ (-\sqrt{a_1}, \sqrt{a_2}, -\sqrt{a_3}, \sqrt{a_4}) \\ C_3 &\ (-\sqrt{a_1}, -\sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}) \\ C_4 &\ (-\sqrt{a_1}, -\sqrt{a_2}, -\sqrt{a_3}, \sqrt{a_4}). \end{aligned} \tag{31}$$

As can easily be verified, the eight invariant points B_i and C_i lie six times four by four on pairs of planes through the six edges of the coordinate tetrahedron; and two by two on four lines through each vertex of the coordinate tetrahedron, such that each pair of planes is harmonically separated by the pair of coordinate planes through the same coordinate edge. For example

$$(\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}), \quad (\sqrt{a_1}, -\sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}),$$
$$(\sqrt{a_1}, -\sqrt{a_2}, -\sqrt{a_3}, \sqrt{a_4}), \quad (\sqrt{a_1}, \sqrt{a_2}, -\sqrt{a_3}, \sqrt{a_4}),$$

lie on a plane α;

$$(-\sqrt{a_1}, \sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}), \quad (-\sqrt{a_1}, -\sqrt{a_2}, \sqrt{a_3}, \sqrt{a_4}),$$
$$(-\sqrt{a_1}, -\sqrt{a_2}, -\sqrt{a_3}, \sqrt{a_4}), \quad (-\sqrt{a_1}, \sqrt{a_2}, -\sqrt{a_3}, \sqrt{a_4}),$$

lie on a plane β such that α, β, $x_1 = 0$, $x_4 = 0$ form a harmonic pencil through A_2A_3. From this follows that $A_1A_2A_3A_4$, $B_1B_2B_3B_4$, $C_1C_2C_3C_4$ form three desmic tetrahedra, or

THEOREM 7. *The twelve points consisting of the four fundamental-points and the eight invariant points of the special cubic transformation form three desmic tetrahedra.*

The equations of α and β are respectively $\sqrt{a_4}x_1 - \sqrt{a_1}x_4 = 0$, $\sqrt{a_4}x_1 + \sqrt{a_1}x_4 = 0$, so that $a_4x_1^2 - a_1x_4^2 = 0$ is a quadric through the eight points. Likewise $a_4x_2^2 - a_2x_4^2 = 0$ and $a_4x_3^2 - a_3x_4^2 = 0$ are such quadrics. They are linearly independent, so that

$$\lambda_1(a_4x_1^2 - a_1x_4^2) + \lambda_2(a_4x_2^2 - a_2x_4^2) + \lambda_3(a_4x_3^2 - a_3x_4^2) = 0$$

represents the net of quadrics through the eight invariant points, so that the polar planes of a point P with respect to the net intersect in the corresponding point P' of the involution.

From this follows that the 28 joins I_iI_k of the eight invariant points B_i and C_i are invariant lines such that *to a point P on I_iI_k corresponds a collinear point P'*, for which two corresponding points P and P' on I_iI_k form a harmonic quadruple with I_i and I_k.

To a bisecant of two opposite edges of the coordinate tetrahedron corresponds a cubic which degenerates into the two edges themselves and another bisecant of the same edges. For example to the bisecant of A_1A_2 and A_3A_4 intersecting A_1A_2 in $C(k_1, k_2, 0, 0)$ and A_3A_4 in $D(0, 0, k_3, k_4)$ corresponds properly the bisecant joining $E(a_1k_2, a_2k_1, 0, 0)$ and $F(0, 0, a_3k_4, a_4k_3)$.

To establish the complex $L(p_{ik}) = 0$ explicitly, the x's must be eliminated from the Plückerian coordinates

$$\rho p'_k = x_j x_l (a_i x_k^2 - a_k x_i^2), \quad i, j, k, l = 1, 2, 3, 4. \tag{32}$$

Making use of the identity connecting the p_{ik}'s, this process of elimination, which for the sake of brevity is omitted, leads to the desired equation of L

$$p_{12}p_{13}p_{23} + p_{12}p_{14}p_{42} + p_{13}p_{14}p_{34} + p_{23}p_{42}p_{34} = 0, \tag{33}$$

which is of degree three as required by the general theory. This complex contains the four special congruences of lines through the vertex, and the three special congruences of lines joining points of opposite edges.

The g_{ik}'s reduce here to seven:

$$\begin{aligned}
\rho g_{12} &= x_3 x_4 (a_1 x_2^2 + a_2 x_1^2) \\
\rho g_{13} &= x_2 x_4 (a_1 x_3^2 + a_3 x_1^2) \\
\rho g_{14} &= x_2 x_3 (a_1 x_4^2 + a_4 x_1^2) \\
\rho g_{23} &= x_1 x_4 (a_2 x_3^2 + a_3 x_2^2) \\
\rho g_{24} &= x_1 x_3 (a_2 x_4^2 + a_4 x_2^2) \\
\rho g_{34} &= x_1 x_2 (a_3 x_4^2 + a_4 x_3^2) \\
\rho g &= a x_1 x_2 x_3 x_4,
\end{aligned} \tag{34}$$

which are linearly independent and which consequently define a rational three-spread in a projective six-space.

In § 13 it will be found that there are 10 linearly independent invariant quartics $Q_i = 0$. Hence by putting $\rho q_i = Q_i$, $i = 1, \cdots, 10$, the couples of the transformation may also be mapped on a rational three-spread in a projective nine-space, so that figures on this spread are mapped back into invariant figures in the space of the cubic transformation.

11. *Invariant Surfaces and Curves of the Cubic Transformation.* In order that a surface of order r may be invariant it is necessary that it pass $r(3-1) = 2r$ times through some or all of the fundamental-points. Thus a plane must pass through two fundamental-points or through an edge of the coordinate tetrahedron and has therefore for example the form $Ax_1 + Bx_2 = 0$. The transformed of this is $Aa_1x_2 + Ba_2x_1 = 0$ and is identical with the given plane when $Ba_2 = \rho A$, $Aa_1 = \rho B$, ρ being a factor of proportionality having the value $\pm\sqrt{a_1a_2}$. There is moreover $\rho A^2 a_1 = \rho B^2 a_2$, or $A^2 = a^2 a_2$, $B^2 = a^2 a_1$, or $A = a\sqrt{a_2}$, $B = a\sqrt{a_1}$. Hence an invariant plane through A_3A_4 has necessarily the form $\sqrt{a_2}x_1 \pm \sqrt{a_1}x_2 = 0$. Similar expressions are found for the invariant planes through the other five edges.

For the two nets of invariant quadrics we find the expression

$$(35)\qquad A(\sqrt{a_3a_4}x_1x_2 \pm \sqrt{a_1a_2}x_3x_4) + B\sqrt{a_2a_4}x_1x_3 \pm \sqrt{a_1a_3}x_2x_4) + C\sqrt{a_2a_3}x_1x_4 \pm \sqrt{a_1a_4}x_2x_3) = 0,$$

determined by the three plus or the three minus signs respectively.

There are two types of invariant cubics due to the fact that from the complete transformed, which is of order nine, coordinate planes must split off by a total of six. This may happen in two ways: first, when the original cubic has three vertices as nodes; second, when the cubic has two of the vertices as nodes and the remaining two as single points. Cubics of this sort are for example

$$(36)\quad A(\sqrt{a_4^3}x_1x_2x_3 \pm \sqrt{a_1a_2a_3}x_4^3) + B(\sqrt{a_3a_4}x_1x_2 \pm \sqrt{a_1a_2}x_3x_4)x_4 + C(\sqrt{a_2a_4}x_1x_3 \pm \sqrt{a_1a_3}x_2x_4)x_4 + D(\sqrt{a_2a_3}x_1x_4 \pm \sqrt{a_1a_4}x_2x_3)x_4 = 0,$$

$$(37)\quad A(a_4x_1x_2x_3 \pm \sqrt{a_1a_2}x_3x_4^2) + B(a_3x_1x_2x_4 \pm \sqrt{a_1a_2}x_3x_4^2) + C(\sqrt{a_2}x_1 + \sqrt{a_1}x_2)x_3x_4 + D(\sqrt{a_2}a_4x_1x_3^2 \pm \sqrt{a_1}a_3x_2x_4^2) + E(\sqrt{a_2}a_3x_1x_4^2 \pm \sqrt{a_1}a_4x_2x_3^2) = 0.$$

A class of invariant quartic surfaces is obtained by means of the linear complex K

$$(38)\qquad \Sigma a_{ik}p_{ik} = 0$$

together with the complex L, (30). They define a congruence of lines through couples of the involution whose locus is the invariant quartic

$$(39)\qquad \Sigma a_{jl}x_ix_k(a_ix_k^2 - a_kx_i^2) = 0, \quad j \neq l$$

having the fundamental-points as double points and passing through the edges A_iA_k, and the eight points $(\pm\sqrt{a_1}, \pm\sqrt{a_2}, \pm\sqrt{a_3}, \pm\sqrt{a_4})$. But the lines joining corresponding points of the invariant quadrics, like $\sqrt{a_2a_4}x_1x_3 \pm \sqrt{a_1a_2}x_3x_4 = 0$, form a special congruence not contained in a linear complex $\Sigma a_{ik}p_{ik} = 0$. Hence for the general invariant quartic surface of the first type terms like $A(a_2a_4x_1^2x_3^2 - a_1a_2x_3^2x_4^2)$ must be added, so that instead of (36) we have more generally

$$(40)\qquad \Sigma a_{ik}x_jx_l(a_ix_k^2 - a_kx_i^2) + \Sigma b_{ik}(a_ja_lx_i^2x_k^2 - a_ia_kx_j^2x_l^2) = 0.$$

For the second type of invariant quartics we obtain

$$(41)\qquad \Sigma a_{ik}x_jx_l(a_ix_k^2 + a_kx_i^2) + \Sigma b_{ik}(a_ja_lx_i^2x_k^2 + a_ia_kx_j^2x_l^2) + cx_1x_2x_3x_4 = 0.$$

In both cases these equations reproduce themselves in the involution multiplied by the factor $a_1a_2a_3a_4x_1^2x_2^2x_3^2x_4^2$. Quartics of these two types are determined by 8 and 9 points (couples) respectively.

Among the first type may in particular be noted the class of Weddle surfaces which are cut out from the complex (30) of the involution by the linear complex

$$(a_1 - a_2)p_{34} + (a_1 - a_3)p_{42} + (a_1 - a_4)p_{23} + (a_2 - a_3)p_{14} + (a_2 - a_4)p_{13} + (a_3 - a_4)p_{12} = 0.$$

The Weddle surface as the locus of couples of the involution belonging to both complexes is thus

$$\begin{aligned}(42)\qquad &(a_1 - a_2)x_1x_2(a_4x_3^2 - a_3x_4^2) + (a_1 - a_3)x_1x_3(a_2x_4^2 - a_4x_2^2) \\ &+ (a_1 - a_4)x_1x_4(a_3x_2^2 - a_2x_3^2) + (a_2 - a_3)x_2x_3(a_4x_1^2 - a_1x_4^2) \\ &+ (a_2 - a_4)x_2x_4(a_1x_3^2 - a_3x_1^2) + (a_3 - a_4)x_3x_4(a_2x_1^2 - a_1x_2^2) = 0.\end{aligned}$$

As is easily verified, it has the vertices of the coordinate tetrahedron, the unit-point, and the point (a_1, a_2, a_3, a_4) as double points and passes moreover through the eight invariant points $(\pm\sqrt{a_1}, \pm\sqrt{a_2}, \pm\sqrt{a_3}, \pm\sqrt{a_4})$.

Another important quartic surface is obtained from Reye's tetrahedral complex

$$(43)\qquad ap_{12}p_{34} + bp_{13}p_{42} + cp_{14}p_{23} = 0,$$

whose lines cut the faces of the coordinate tetrahedron in points of a constant cross-ratio, as is well-known. The locus of couples of corresponding points on lines of this complex is

$$\begin{aligned}(44)\qquad &a(a_1x_1^2 - a_2x_2^2)(a_4x_3^2 - a_3x_4^2) + \\ &b(a_3x_1^2 - a_1x_3^2)(a_2x_4^2 - a_4x_2^2) + \\ &c(a_4x_1^2 - a_1x_4^2)(a_3x_2^2 - a_2x_3^2) = 0,\end{aligned}$$

a desmic quartic whose nodes are the twelve points A_i, B_i, C_i, $i = 1, 2, 3, 4$. Hence

THEOREM 8. *The locus of couples of corresponding points of the special cubic transformation on lines of Reye's tetrahedral complex is a desmic surface whose desmic tetrahedra are formed by the fundamental-points and invariant points of the transformation. The sixteen joins of corresponding points in the perspectives between the three desmic tetrahedra lie on the desmic surface.*

12. *The Cubic Transformation and the Cayley Cubic.* To the plane

$$p = b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 = 0 \tag{45}$$

corresponds the Cayley cubic

$$S_3 = a_1b_1x_2x_3x_4 + a_2b_2x_1x_3x_4 + a_3b_3x_1x_2x_4 + a_4b_4x_1x_2x_3 = 0. \tag{46}$$

The plane cuts the coordinate tetrahedron in a quadrilateral with the six vertices $\mathcal{E}_{12}$, $\mathcal{E}_{13}$, $\mathcal{E}_{14}$, $\mathcal{E}_{34}$, $\mathcal{E}_{42}$, $\mathcal{E}_{23}$ to whose diagonal triangle $\mathcal{E}_{12}\mathcal{E}_{34}$, $\mathcal{E}_{13}\mathcal{E}_{42}$, $\mathcal{E}_{14}\mathcal{E}_{23}$ corresponds the diagonal triangle of the quadrilateral cut out from the coordinate tetrahedron by the plane

$$a_1b_2b_3b_4x_1 + a_2b_1b_3b_4x_2 + a_3b_1b_2b_4x_3 + a_4b_1b_2b_3x_4 = 0. \tag{47}$$

This same plane cuts the Cayley cubic in the latter diagonal triangle, as may easily be verified. Thus we have incidentally (choosing $a_1 = a_2 = a_3$)

THEOREM 9. *The plane* $b_2b_3b_4x_1 + b_1b_3b_4x_2 + b_1b_2b_4x_3 + b_1b_2b_3x_4 = 0$ *which is obtained by interchanging the b's and x's in the cubic* $b_1x_2x_3x_4 + b_2x_1x_3x_4 + b_3x_1x_2x_4 + b_4x_1x_2x_3 = 0$ *cuts this cubic in three lines.*

To the pencil of planes $\Sigma b_ix_i + \lambda x_1 = 0$ corresponds the pencil of Cayley cubics $\Sigma a_ib_ix_jx_kx_l + \lambda x_2x_3x_4 = 0$, with common tangent planes $a_3b_3x_2 + a_2b_2x_3 = 0$, $a_4b_4x_3 + a_3b_3x_4 = 0$, $a_2b_2x_4 + a_4b_4x_2 = 0$ along the edges A_1A_4, A_1A_2, A_1A_3.

A quadric Q cuts the Cayley cubic S_3 in a sextic C_6 of genus four. To Q corresponds in the involution a sextic surface S_6 with the edges A_iA_k as nodal lines. S_6 cuts the plane p in a sextic curve C_6 with double points at each of the six $\mathcal{E}_{ik}$'s, (Wirtinger sextic).* As the C_6 is the most general space sextic of genus four, since such a sextic admits of 255 Cayley cubics through it, we have

THEOREM 10. *A general space sextic of genus four may be transformed in 255 different ways into plane Wirtinger sextics.*

The following considerations lead to an interesting theorem on linear point series on a Wirtinger sextic. For the sake of simplicity assume again $a_1 = a_2 = a_3$. Then to the unit-plane $p = x_1 + x_2 + x_3 + x_4 = 0$ corresponds the Cayley cubic $S_3 = x_2x_3x_4 + x_1x_3x_4 + x_1x_2x_4 + x_1x_2x_3 = 0$. Putting $\phi_1 = p$, $\phi_2 = \Sigma x_ix_k$, the symmetric quadric $\phi_1^2 + \lambda\phi_2 = 0$ is uniquely determined (i. e., λ) by the condition that it shall pass through a generic point P.

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* *Mathematische Annalen*, Vol. 40 (1892), pp. 261-312.

If P lies on Q then there are 23 other points on Q which with P belong to the symmetric group G_{24}. Now if P lies on the sextic C_6, the intersection of Q and S_3, then all 24 points of the G_{24} lie on C_6. As is well known,* these 24 points lie in sixes four times on four conics whose planes pass by fours through the four lines in which the planes of the coordinate tetrahedron cut the unit-plane. Thus the 24 points lie for example on four planes through $\mathcal{E}_{12}$, $\mathcal{E}_{23}$, $\mathcal{E}_{31}$. To these correspond four Cayley cubics which touch each other along A_4A_1, A_4A_2, A_4A_3, with the common tangent planes $x_2 + x_3 = 0$, $x_3 + x_4 = 0$, $x_1 + x_2 = 0$. The four planes cut the Cayley cubic S_3 in four cubics to which correspond in p four plane cubics of a pencil, passing through all six $\mathcal{E}_{ik}$'s and touching each other at $\mathcal{E}_{41}$, $\mathcal{E}_{42}$, $\mathcal{E}_{43}$. An analogous result is obtained for the remaining three quadruples of planes containing the 24 points. As every point P on C_6 gives rise to such a configuration we may state

THEOREM 11. *On a Wirtinger sextic associated with the symmetric group of order 24, in the manner described, there are ∞^1 sets of 24 points each, such that each set lies by sixes four times on four cubics of a pencil through the six double points. The cubics of a pencil touch each other at three of the double points and pass singly through the remaining ones.*

quartics through the coordinate tetrahedron cutting the S_3 in distinct sextics. be established. The 24 points on the C_6 also lie on a symmetric quartic surface $\phi_1^4 + \lambda\phi_1^2\phi_2 + \mu\phi_1\phi_3 + \nu\phi_2^2 + \omega\phi_4 = 0$ which cuts the S_3 in a curve of order 12. To the quartic surface corresponds a surface of order 12 cutting p in a curve C_{12} of order 12 with quadruple points at the $\mathcal{E}_{ik}$'s. The C_{12} cuts the C_6 in 72 points of which $6 \cdot 2 \cdot 4 = 48$ are absorbed at the $\mathcal{E}_{ik}$'s; hence there remains a residual set of 24 points which are precisely the ones corresponding to the G_{24} on C_6. As P lies on the Cayley cubic $\phi_3 = 0$, and also on the quadric $\phi_1^2 + \lambda\phi_2 = 0$, there is one symmetric quartic surface through the point P and the remainder of the G_{24}, i. e., $\phi_2^2 + k\phi_4 = 0$, k being determined by P.

13. *The Schur Sextic.* Consider again a plane p and the corresponding Cayley cubic S_3 with the coordinate tetrahedron as the nodal tetrahedron. S_3 cuts p in an invariant plane cubic C_3 through the six points $\mathcal{E}_{ik}$ in p.

* Emch, "Some Geometric Applications of Symmetric Substitution Groups," *American Journal of Mathematics*, Vol. XLV, pp. 192-207 (1923).

A quartic surface S_4 through the edges of the coordinate tetrahedron depends on $34 - 22 = 12$ constants. The S_3 and an S_4 cut in a sextic curve C_6 of genus three. If a is any plane, then $S_3 a + \lambda S_4 = 0$ represents a system of ∞^4 quartics through C_6 and the coordinate tetrahedron. Hence there are ∞^8 quartics through the coordinae tetrahedron cutting the S_3 in distinct sextics. As there are ∞^{12} tetrahedra and thrugh each of these ∞^3 Cayley cubics S_3, there are $\infty^{12+3+8} = \infty^{23}$ sextics C_6 as residual curves of intersection of cubic and quartic surfaces through the tetrahedron. A sextic C_6 of this system passes through the vertices of the tetrahedron of the S_3 and the S_4 from which it is generated and cuts each of the six edges in an additional point, as is easily ascertained. The edges of the tetrahedron are therefore trisecants of the C_6.

In the involutory cubic transformation associated with the C_6, to each edge of the tetrahedron corresponds a point on C_6, to each vertex a trisecant of C_6; thus to the vertices and edges of the tetrahedron correspond a quadrilateral and its six vertices which lie on the C_6. The sides of the quadrilateral are trisecants of the C_6. This is precisely the sextic investigated by F. Schur,* depending on 23 constants as compared with 24 in the general case.

In our special cubic transformation associated with the tetrahedron, to the S_3 corresponds a plane p, to S_4 a quartic which cuts p in a plane quartic C_4 through the $\mathcal{E}_{ik}$'s. This plane quartic, being general, is of genus three and corresponds in the special involution to the sextic C_6.

Conversely, if we pass a plane quartic C_4 through the $\mathcal{E}_{ik}$'s in p we dispose of 14 constants of which 6 are absorbed by the $\mathcal{E}_{ik}$'s. A quartic surface S_4, through the tetrahedron of the transformation depends on 12 constants, of which we may choose 8 arbitrarily as points in p determining the C_4 in p. Thus there are ∞^4 quartic surfaces S_4 through C_4 and the edges of the tetrahedron which intersect in the same residual curve C_6 on the Cayley cubic S_3 corresponding to p. The results obtained may be stated as

THEOREM 12. *Let p be a plane and S_3 the Cayley cubic corresponding to it in the special cubic transformation associated with the tetrahedron $A_1A_2A_3A_4$ whose edges cut p in six points $\mathcal{E}_{ik}$. To a plane quartic C_4 through the $\mathcal{E}_{ik}$'s corresponds a sextic C_6 on the Cayley cubic for which the edges of the tetrahedron are trisecants. The same sextic is cut out by ∞^4 quartics S_4 through the edges of the tetrahedron and C_4. All sextics of this kind*

* "Ueber die durch collineare Grundgebilde erzeugten Curven und Flächen," *Mathematische Annalen*, Vol. 18, pp. 1-32 (1881).

form an ∞^{23} manifold and are characterized by the property that with each C_6 is associated a plane which cuts it in six vertices of a quadrilateral.

As F. Schur seems to have been the first to discover this possibility, these curves might properly be called Schur sextics.

In the same paper Schur also found a class of ∞^{20} sextics of genus three with ∞^1 tetrahedra of trisecants.

14. *The Enriquez Sextic Surface.* In an important memoir,* Enriquez gives an example of a sextic surface S_6

$$\phi_2(x_2x_3x_4,\ x_1x_3x_4,\ x_1x_2x_4,\ x_1x_2x_3) + x_1x_2x_3x_4\ f_2(x_1,\ x_2,\ x_3,\ x_4) = 0, \tag{48}$$

which has the vertices and edges of the coordinate tetrahedron as triple points and nodal lines respectively. As ϕ and f are quadratic in $x_2x_3x_4, \cdots$, and $x_1, \cdots$ respectively this fact is obvious from the equation. The point which Enriquez makes is that this is a surface with both the geometric and arithmetical genus equal to zero but with the bigenus $P = 1$, or a surface on which the double of a linear system $| C |$ of curves is contained in the double of the adjoint system $| C' |$. The adjoint system $| C' |$ of the system of plane sections $| C |$ of the sextic is cut out by the Cayley cubics through the coordinate tetrahedron. $| C' |$ does not contain $| C |$. On the other hand the double $| 2C |$ is contained in that of $| 2C' |$.

Castelnuovo † proves that the necessary and sufficient condition for a rational surface is that arithmetic genus and bigenus are zero. The geometric genus zero is a consequence of the bigenus being zero, but not conversely.

The linear systems which are used in these investigations are brought out in a particularly clear light by means of the special cubic transformation.

In the first place, to an Enriquez sextic S_6 corresponds a sextic S_6' of the same type. To the ∞^3 systems of planes cutting from S_6 the system $| C |$ of plane sextics of genus four corresponds the ∞^3 systems of Cayley cubics cutting S_6' in the system $| C' |$ of space sextics of the same genus. The system $| 2C |$ is contained in the system of curves of order 12 cut out on S_6 by the system of all quadrics f_2. To it corresponds on S_6' the system of duo- to the system of quadrics. This system of dudecics is contained in the system decics cut out by the special system of sextics $\phi_2(x_2x_3x_4, \cdots)$ corresponding

* "Introduzione alla geometria sopra le superficie algebriche," *Societa Italiana della Scienze*, series 3, Vol. 10, p. 66 (1896).

† "Sulle superficie di genere zero," *Soieta Italiana delle Science*, series 3, Vol. 10, pp. 120-3 (1896).

$|\, 2C' \,|$ of duodecics cut out on S_6' by the system of Enriquez sextics. Thus the systems cut out on S_6 by the quadrics and sextics are equivalent.

Noticing that in the transformation a quadric f_2 is transformed into a special Enriquez sextic ϕ_2, and conversely, it is evident that in the case of proportionality of the coefficients of f_2 and ϕ_2 the sextic

$$\phi_2 + x_1x_2x_3x_4\ f_2 = 0$$

is invariant and depends on 10 effective constants, assuming the transformation in the simple form where $a_1 = a_2 = a_3 = a_4$. The duodecics of the system of intersections of such sextic surfaces are, of course, also invariant as well as those of corresponding sextics S_6 and S_6'. We have thus

THEOREM 13. *The system of ∞^{13} Enriquez sextics is invariant in the special cubic transformation. Among the system are ∞^{10} invariant sextics.*

The system of ∞^3 symmetric Enriquez sextics may be written in the form

$$\lambda_0(x_2^2x_3^2x_4^2 + \cdots) + \lambda(x_1x_2x_3^2x_4^2 + \cdots) + x_1x_2x_3x_4\,\{\mu(x_1^2 + \cdots) + \nu(x_1x_2 + \cdots)\} = 0.$$

To it corresponds in the transformation the system

$$x_1x_2x_3x_4\,\{(x_1^2 + \cdots) + \lambda(x_1x_2 + \cdots)\} + \mu(x_2^2x_3^2x_4^2 + \cdots) + \nu(x_1x_2x_3^2x_4^2 + \cdots) = 0.$$

Two sextics of this kind intersect in a duodecic as before. If each equation is divided through by the coefficient of the first parenthesis, then their difference, after dividing through by $x_1x_2x_3x_4$, reduces to a quadric. Hence such duodecics lie on quadrics. Hence also a symmetric sextic and its corresponding sextic intersect in an invariant duodecic C_{12} which lies on a symmetric quadric.

The C_{12} is invariant in the symmetric group G_{24} of collineations. There are ∞^1 sets of 24 points $P_1, \cdots, P_{24}$ each of which is invariant in the G_{24}. But as the C_{12} is also invariant in the cubic transformation T, the latter transforms $P_1, \cdots, P_{24}$ into 24 new points $P'_1, \cdots, P'_{24}$ on C_{12}. Any of the P_i' is obtained from any of the P_i by the product of a substitution S of the G_{24} and T, and conversely. Hence there exists a group of order 48 of Cremona transformations which leave the C_{12} invariant. Thus

THEOREM 14. *The system of duodecics obtained by the intersections of symmetric Enriquez sextics and their respective cubic transforms is invariant in a group of Cremona transformations of order 48.*

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The Invariant System of two Associated Bilinear Connexes.*

By Oliver E. Glenn.

Systems of invariants of connexes were studied by Clebsch and Gordan who published, jointly, a paper on the theory † in 1869. The subject of complete systems remained stationary until 1915 when the present writer developed an algorithm of ternary transvection and of the system derived by transvection ‡ between two finite and complete systems of connexes. A transvectant of four indices of two general connexes,

$$(1)\qquad \begin{aligned} \phi &= A a_{1x} a_{2x} \cdots a_{rx} a_{1u} a_{2u} \cdots a_{su}, \\ \psi &= B b_{1x} b_{2x} \cdots b_{\rho x} \beta_{1u} \beta_{2u} \cdots \beta_{\sigma u}, \end{aligned}$$

the notation for which is,

$$(2)\qquad \tau = (\phi, \psi)^{\,i\ \ j}_{k\ \ l}\ ,$$

is the foundation of this theory. The variables u_1, u_2, u_3 in these formulas are contragredient to x_1, x_2, x_3, and the usual notation for symbolical linear forms is followed, viz.,

$$(3)\qquad \begin{aligned} a_{ix} &= a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3, \\ a_{iu} &= a_{i1}u_1 + a_{i2}u_2 + a_{i3}u_3. \end{aligned}$$

Among the applications that have been made of the transvection process is the determination, included in my article which is quoted above, of the fundamental simultaneous system of a conic p_x^2 and a bilinear connex, $a_x a_u$.

The object of the present paper is to give the results of a research which solved the problem of rank next to the above in degree of difficulty, that is, the determination of a fundamental system of invariant formations of all categories of the set consisting of two bilinear connexes,

$$(4)\qquad f = a_x a_u = b_x \beta_u = \cdots, \quad \phi = p_x \theta_u = q_x \iota_u = \cdots.$$

The geometry which is naturally associated with this problem is perhaps

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* Presented to the American Mathematical Society, May, 1925.

† *Math. Annalen*, 1 (1869), 359.

‡ *Trans. Amer. Math. Soc.*, 17 (1916), 405.

simpler and certainly more symmetrical than that of one connex taken with a conic.* To every point (x) there corresponds two points (f), (ϕ). To every line (u) there corresponds two lines $\{f\}$, $\{\phi\}$. Thus the basic configuration is a three-line and a three-point, a perfect duality.

Consequently we have been able to reduce an unusually large number of forms which belonged originally to the system under consideration. On the general significance of connexes in a plane, also, we may quote as follows from the before-mentioned paper of Clebsch and Gordan: "Ein genauerer Einblick in die Theorie der ternären algebraischen Formen lehrt, dass gewisse allgemeine Eigenschaften erst dann vollständig erkannt werden können, wenn man eine Form von der Gestalt einer Zwischenform zur Grundform wählt."

The latter authors determined the system of $f = a_x a_u$ and, from their article, we may write, as known, the following finite and complete individual systems for f and ϕ, respectively,

$$(5)\quad \begin{aligned} &f,\ f_1 = a_x\beta_u b_a, \quad && g = b_a a_x c_x(\beta\gamma x), && h = b_a\beta_u\gamma_u(acu),\\ &\qquad i = a_a, && i_1 = a_\beta b_a, && i_2 = a_\beta b_\gamma c_a. \end{aligned}$$

$$(6)\quad \begin{aligned} &\phi,\ \phi_1 = p_x\iota_u q_\theta, \quad && \sigma = q_\theta p_x r_x(\iota\kappa x), && \tau = q_\theta\iota_u\kappa_u(pru),\\ &\qquad \epsilon = p_\theta, && \epsilon_1 = p_\iota q_\theta, && \epsilon_2 = p_\iota q_\kappa r_\theta. \end{aligned}$$

THE ALGORITHM OF TERNARY TRANSVECTION. The true and complete algorithm of transvection between two ternary mixed concomitants (Zwischenformen) was discovered by myself and described in the paper mentioned above. It consists of the following operations. Polarize ϕ, (cf. (1)), by

$$(7)\quad \Sigma\left(y_1^{(1)}\frac{\partial}{\partial x}\right)^{\iota_1}\left(y_2^{(1)}\frac{\partial}{\partial x}\right)^{\iota_2}\cdots\left(y_\sigma^{(1)}\frac{\partial}{\partial x}\right)^{\iota_\sigma}\left(y_1^{(2)}\frac{\partial}{\partial x}\right)^{\kappa_1}\left(y_2^{(2)}\frac{\partial}{\partial x}\right)^{\kappa_2}\cdots\left(y_\rho^{(2)}\frac{\partial}{\partial x}\right)^{\kappa_\rho}$$
$$\times\left(v_1^{(1)}\frac{\partial}{\partial u}\right)^{\lambda_1}\left(v_2^{(1)}\frac{\partial}{\partial u}\right)^{\lambda_2}\cdots\left(v_\rho^{(1)}\frac{\partial}{\partial u}\right)^{\lambda_\rho}\left(v_1^{(2)}\frac{\partial}{\partial u}\right)^{\mu_1}\left(v_2^{(2)}\frac{\partial}{\partial u}\right)^{\mu_2}\cdots\left(v_\sigma^{(2)}\frac{\partial}{\partial u}\right)^{\mu_\sigma},$$

where,

$$(8)\qquad \left(y_o^{(e)}\frac{\partial}{\partial x}\right) = y_{1o}^{(e)}\frac{\partial}{\partial x_1} + y_{2o}^{(e)}\frac{\partial}{\partial x_2} + y_{3o}^{(e)}\frac{\partial}{\partial x_3}\ .$$

The terms of the summation are uniquely given by the sets of solutions of the restricted system of linear diophantine equations,

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* Clebsch, *Vorlesungen über Geometrie*, (1876), 924.

$$(9) \qquad \sum_{t=1}^{\sigma} \iota_t = i, \quad \sum_{\tau=1}^{\rho} \kappa_\tau = j, \quad \sum_{\tau=1}^{\rho} \lambda_\tau = k, \quad \sum_{t=1}^{\sigma} \mu_t = l.$$

These are subject to conditions,

$$i + j \leqq r, \quad k + l \leqq s, \quad i + l \leqq \sigma, \quad j + k \leqq \rho,$$

and are to be satisfied only by values 0, 1 of the variables, i, j, k, l being fixed chosen integers. With t, τ assigned at least one number in each pair (ι_t, μ_t), $(\kappa_\tau, \lambda_\tau)$ must be zero.

We substitute in the resulting polar as follows:

$$(10) \qquad \begin{aligned} &y_p^{(1)} = \beta_p, \ (p = 1, \cdots, \sigma); \quad y_p^{(2)} = (b_p u), \ (p = 1, \cdots, \rho), \\ &v_p^{(1)} = b_p, \ (p = 1, \cdots, \rho); \quad v_p^{(2)} = (\beta_p x), \ (p = 1, \cdots, \sigma). \end{aligned}$$

Finally, multiply each term of the result by the b_x, β_u factors (of ψ) which are not then involved in it. The resulting concomitant * is the transvectant,

$$(11) \qquad \tau = (\phi, \psi)_k^{i\ j}{}_l \ .$$

RÉSUMÉ OF METHODS. If two infinite systems, $[\phi]$, $[\psi]$, of connexes of the type (1), both have the technical properties of finiteness and completeness then the system, $[C]$, derived by transvection between $[\phi]$ and $[\psi]$ is finite and complete.

By $[C]$ is meant the system which consists of all terms of all transvectants $\tau = (\phi, \psi)_k^{i\ j}{}_l$, where ϕ is a form chosen from $[\phi]$ and ψ is a form chosen from $[\psi]$. We can frequently use, instead of a term of a transvectant, however, the whole transvectant, in view of a theorem that the difference between a transvectant and one of its terms is a sum of terms of transvectants of lower grade number,

$$(12) \qquad g = r + \rho + s + \sigma,$$

of forms obtained from ϕ and ψ, respectively, by convolution.

The principal theorem of practical use in the problem of finding a complete simultaneous system by transvection between two known finite and com-

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* The relation of this process to the work of former authors is carefully explained in my previous paper. The processes of convolution involved were used by nearly all previous writers.

plete systems, such as (5), (6) above, is the following: Let the matrix of four indices of τ be partitioned into two matrices,

$$(13) \qquad \begin{pmatrix} i & j \\ k & l \end{pmatrix} = \begin{pmatrix} i_1 + i_2 & j_1 + j_2 \\ k_1 + k_2 & l_1 + l_2 \end{pmatrix} = \begin{pmatrix} i_1 & j_1 \\ k_1 & l_1 \end{pmatrix} + \begin{pmatrix} i_2 & j_2 \\ k_2 & l_2 \end{pmatrix},$$

and suppose that,

$$(14) \qquad \phi = \phi_1\phi_2, \qquad \psi = \psi_1\psi_2,$$

while both of the following transvectants exist, have a non-symbolical interpretation in terms of the actual coefficients and variables of the original forms and are not zero:

$$\tau_1 = (\phi_1, \psi_1)^{\,i_1\ j_1}_{\,k_1\ l_1} \qquad \tau_2 = (\phi_2, \psi_2)^{\,i_2\ j_2}_{\,k_2\ l_2}\ .$$

Then τ contains reducible terms and since, in the actual work of identifying a minimum complete system we arrange our transvectants in a linear sequence according to ascending grade, τ may be omitted from a complete system. We shall refer to this process as the reduction of τ by separation.

REDUCTION BY MEANS OF IDENTITIES. It is generally true that all transvectants represented by the system $[C]$, derived by transvection between two finite and complete systems, excepting a finite number of these transvectants, may be reduced by separation.

Within the finite system generated thus, however, it is possible to make some additional reductions by identities. The following are fundamental for the problem under consideration.

Determinant products. The formula,

$$(15) \qquad (abu)(\alpha\beta x) = \begin{vmatrix} a_\alpha & a_\beta & a_x \\ b_\alpha & b_\beta & b_x \\ a_u & \beta_u & u_x \end{vmatrix},$$

usually reduces a monomial which contains $(abu)(\alpha\beta x)$ as a factor.

Syzygies. In the system $[C]$ all transvectants τ in which the form ϕ contains gh as a factor are reducible on account of the syzygy,

$$(16) \qquad gh = \begin{vmatrix} i''f - ii''u_x + i'f_2 & i''u_x & f_2 \\ i''u_x & if - i'u_x + f_2 & f \\ f_2 & f & u_x \end{vmatrix}$$
$$= S(f, f_1, i, i_1, i_2, u_x),$$

where,

$$i' = \tfrac{1}{2}(i^2 - i_1), \quad i'' = \tfrac{1}{6}(i^3 - 3ii_1 + 2i_2), \quad f_2 = f_1 - if + i'u_x.$$

For the same reason τ is reducible if ψ contains the factor,

$$\sigma\tau = S(\phi, \phi_1, \epsilon, \epsilon_1, \epsilon_2, u_x). \tag{17}$$

Linkages. The following is an identity if $a_x a_u = b_x \beta_u = c_x \gamma_u$:

$$a_x b_a c_\beta \gamma_u = if_1 - i'f + i''u_x. \tag{17$_1$}$$

The Finite System of Transvectants. The system which is derived by transvection from the systems (5), (6), is given by the formula,

$$T = (f^\mu f_1^{\ \nu} g^m h^n, \quad \phi^{\mu'} \phi_1^{\ \nu'} \sigma^{m'} \tau^{n'})\, {}^{i}_{k}\ {}^{j}_{l}\ , \tag{18}$$

and the six invariants of the two sets. The exponents of the forms in T may have any positive integral values, thus an infinitude of transvectants is represented. The initial problem is, therefore, to list all transvectants of the formula (18) which are not reducible by separation. This was done by arranging them according to ascending grade and examining them one at a time, all of grade 4 first, then all of grade 5 and so forth. When the grade 12 was reached it became clear that all of higher grade were reducible by separation. In fact the highest grade for which there exists an irreducible transvectant is 12.

The number of transvectants (18) which are irreducible by the process of separation or by means of the syzygies (16), (17) is 218.

I have not included all of these cases in the final tables below since 40 transvectants proved to be reducible by identities.

It has not seemed desirable, either, to give details of the reduction of all of these 40. Some of the typical reductions,* however, will now be considered.

Typical Cases of Reduction. The transvectant of grade 5, $(f, \sigma)^{01}_{10}$, has the term,

$$t = q_\theta r_a\ (apu)(\kappa x) = q_\theta r_a \begin{vmatrix} a_\iota & a_\kappa & a_x \\ p_\iota & p_\kappa & p_x \\ \iota_u & \kappa_u & u_x \end{vmatrix} \tag{19}$$

$$= a_\iota p_\kappa q_\theta r_a u_x + p_\iota q_\theta r_a \kappa_u a_x + a_\kappa q_\theta r_a \iota_u p_x$$
$$- p_\kappa q_\theta r_a \iota_u a_x - a_\iota q_\theta r_a \kappa_u p_x - a_\kappa p_\iota q_\theta r_a u_x \equiv \sum_{i=1}^{6} t_i.$$

* W. Sensenig, *Amer. Journal of Math.*, 41 (1919), 111.

The first one of the latter six monomials is,

$$t_1 = u_x(r_x p_\kappa q_\theta \iota u, a_x a_u)\,{}^{10}_{10},$$

The first form in this transvectant is a linkage (17_1). Hence, aside from the factor u_x, t_1 is a sum of terms, each containing a pure invariant as a factor. Likewise,

$$t_4 = -(r_x p_\kappa q_\theta \iota u, a_x a_u)\,{}^{10}_{00}.$$

is reducible. The form $-t_5$ is a term of,

$$\tau = (q_\theta p_{x\iota u} r_{x\kappa u}, a_x a_u)\,{}^{10}_{10},$$

and τ has a reducible term $\tau_1 = a_\iota p_\alpha q_{\theta\kappa u} r_x$. Then,

$$\tau_1 - (-t_5) = a_\iota q_{\theta\kappa u}(p_\alpha r_x - p_x r_\alpha) = a_\iota q_{\theta\kappa u}(\overline{pra}x). \tag{20}$$

Since the latter monomial is a term of a transvectant of grade five which is retained in the fundamental system, viz., $-(f,\tau)\,{}^{10}_{01}\ t_5$ is reduced according to the formula,

$$t_5 = -\tau_1 + \Sigma\tau', \tag{21}$$

where $\Sigma\tau'$ is a sum of terms of a form retained in the table and terms of other transvectants of grade < 5.

The following reductions are evident:

$$\begin{aligned} t_2 &= \epsilon_1(f,\phi)\,{}^{00}_{10}, \\ t_3 &= \phi_1(f,\phi)\,{}^{10}_{10}, \\ t_6 &= -\epsilon_1(f,\phi)\,{}^{10}_{10}\ u_x. \end{aligned} \tag{22}$$

Hence $(f,\sigma)\,{}^{01}_{10}$ has a reducible term t, and is, therefore, itself reduced, according to the general theory.

A different procedure is required for $(h,\phi\tau)\,{}^{00}_{12}$ which has the term,

$$t = b_\alpha p_\beta r_\iota \lambda_u (qsu)(\gamma\theta x)(\overline{ac\kappa}x) \tag{23}$$

$$= b_\alpha p_\beta r_\iota \lambda_u \begin{vmatrix} \begin{vmatrix} a_\gamma & a_\theta & a_x \\ c_\gamma & c_\theta & c_x \\ q_\gamma & q_\theta & q_x \end{vmatrix} & \begin{vmatrix} a_\gamma & a_\theta & a_x \\ c_\gamma & c_\theta & c_x \\ s_\gamma & s_\theta & s_x \end{vmatrix} & \begin{vmatrix} a_\gamma & a_\theta & a_x \\ c_\gamma & c_\theta & c_x \\ \gamma_u & \theta_u & u_x \end{vmatrix} \\ q_\kappa & s_\kappa & \kappa_u \\ q_x & s_x & u_x \end{vmatrix}.$$

When this determinant is expanded its 36 terms are all readily seen to be reducible.

DESCRIPTION OF THE TABLES. A fundamental system of invariant formations of f and ϕ, regarded as being transformed simultaneously by linear transformations, is composed of the 178 transvectants tabulated below, together with the forms (5), (6). A fundamental system is also obtained by taking the one term indicated of each transvectant. The reader is reminded that the equalities in the table are not numerical but congruencial and in each case are to be read *modulis* terms of transvectants of grade lower than that of the transvectant under consideration.

The concomitants may be expressed in terms of the actual coefficients and variables by use of the identities,

$$f = a_x a_u = \sum_{i=1}^{3} (a_{i1}x_1 + a_{i2}x_2 + a_{i3}x_3)u_i,$$

$$\phi = p_x \theta_u = \sum_{i=1}^{3} (b_{i1}x_1 + b_{i2}x_2 + b_{i3}x_3)u_i. \tag{24}$$

Another notation, much used in tensor analysis, is,

$$f = a_{ij}u_i x_j, \quad g = b_{ij}u_i x_j. \tag{25}$$

Equivalent symbols used in the fundamental concomitants comprise the following sets,

$$a_x a_u = \cdots = f_x \zeta_u, \quad p_x \theta_u = \cdots = t_x \mu_u = v_x v_u, \tag{26}$$

departure from the alphabetical order having been made by use of the symbol v instead of u, for obvious reasons.

An important general reference on the subject of ternariants is Study's *Methoden zur Theorie der ternären Formen* (1889).

The Fundamental System.

GRADE 4.

$$(f, \phi)^{10}_{00} = a_\theta a_u p_x \qquad (f, \phi)^{01}_{00} = (apu) a_u \theta_u$$

$$(f, \phi)^{00}_{10} = p_a a_x \theta_u \qquad (f, \phi)^{00}_{01} = (a\theta x) a_x p_x$$

$$(f, \phi)^{10}_{10} = a_\theta p_a \qquad (f, \phi_1)^{10}_{00} = a_\iota q_\theta a_u p_x$$

$$(f, \phi_1)^{01}_{00} = q_\theta (apu) a_u \iota_u \qquad (f, \phi_1)^{00}_{10} = p_a q_\theta a_x \iota_u$$
$$(f, \phi_1)^{00}_{01} = q_\theta (a\iota x) a_x p_x \qquad (f, \phi_1)^{10}_{10} = a_\iota p_a q_\theta$$
$$(f_1, \phi)^{10}_{00} = a_\theta b_a \beta_u p_x \qquad (f_1, \phi)^{01}_{00} = b_a (apu) \beta_u \theta_u$$
$$(f_1, \phi)^{00}_{10} = b_a a_x p_\beta \theta_u \qquad (f_1, \phi)^{00}_{01} = b_a a_x (\beta\theta x) p_x$$
$$(f_1, \phi)^{10}_{10} = a_\theta b_a p_\beta \qquad (f_1, \phi_1)^{10}_{00} = b_a a_\iota q_\theta \beta_u p_x$$
$$(f_1, \phi_1)^{01}_{00} = b_a q_\theta (apu) \beta_u \iota_u \qquad (f_1, \phi_1)^{00}_{10} = b_a p_\beta q_\theta \iota_u a_x$$
$$(f_1, \phi_1)^{00}_{01} = b_a q_\theta (\beta\iota x) a_x p_x \qquad (f_1, \phi_1)^{10}_{10} = a_\iota b_a p_\beta q_\theta$$

GRADE 5

$$(f, \sigma)^{00}_{10} = p_a q_\theta a_x r_x (\iota\kappa x) \qquad (f, \tau)^{10}_{00} = a_\iota q_\theta a_u \kappa_u (pru)$$
$$(f, \tau)^{10}_{01} = a_\iota q_\theta (pru)(a\kappa x) \qquad (f_1, \sigma)^{00}_{10} = b_a p_\beta q_\theta a_x r_x (\iota\kappa x)$$
$$(f_1, \tau)^{10}_{00} = a_\iota b_a q_\theta \beta_u \kappa_u (pru) \qquad (f_1, \tau)^{10}_{01} = a_\kappa b_a q_\theta (pru)(\beta\iota x)$$
$$(g, \phi)^{10}_{00} = a_\theta b_a c_x p_x (\beta\gamma x) \qquad (g, \phi_1)^{10}_{00} = a_\iota b_a q_\theta c_x p_x (\beta\gamma x)$$
$$(h, \phi)^{00}_{10} = b_a p_\beta \gamma_u \theta_u (acu) \qquad (h, \phi)^{00}_{11} = b_a p_\gamma (acu)(\beta\theta x)$$
$$(h, \phi_1)^{00}_{10} = b_a p_\beta q_\theta \gamma_u \iota_u (acu) \qquad (h, \phi_1)^{00}_{11} = b_a p_\beta q_\theta (acu)(\gamma\iota x)$$

GRADE 6

$$(f, \phi^2)^{10}_{01} = a_\theta p_x q_x (a\iota x) \qquad (f, \phi^2)^{01}_{10} = q_a \theta_u \iota_u (apu)$$
$$(f, \phi\phi_1)^{10}_{01} = a_\theta r_\iota p_x q_x (a\kappa x) \qquad (f, \phi\phi_1)^{01}_{10} = q_a r_\iota \theta_u \kappa_u (apu)$$
$$(f, \phi_1^2)^{10}_{01} = a_\iota q_\theta s_\kappa p_x r_x (a\lambda x) \qquad (f, \phi_1^2)^{01}_{10} = q_\theta r_a s_\kappa \iota_u \lambda_u (apu)$$
$$(f_1, \phi^2)^{10}_{01} = a_\theta b_a p_x q_x (\beta\iota x) \qquad (f_1, \phi^2)^{01}_{10} = b_a q_\beta \theta_u \iota_u (apu)$$
$$(f_1, \phi\phi_1)^{10}_{01} = a_\theta b_a r_\iota p_x q_x (\beta\kappa x) \qquad (f_1, \phi\phi_1)^{01}_{10} = b_a q_\beta r_\iota \theta_u \kappa_u (apu)$$
$$(f_1, \phi_1^2)^{10}_{01} = a_\iota b_a q_\theta s_\kappa p_x r_x (\beta\lambda x) \qquad (f_1, \phi_1^2)^{01}_{10} = b_a q_\theta r_\beta s_\kappa \iota_u \lambda_u (apu)$$
$$(g, \sigma)^{03}_{00} = b_a q_\theta (apu)(cru)(\overline{\beta\gamma}\,\overline{\iota\kappa}\,u) \qquad (g, \tau)^{20}_{00} = a_\iota b_a c_\kappa q_\theta (pru)(\beta\gamma x)$$
$$(g, \tau)^{30}_{00} = b_a c_\kappa q_\theta (apr)(\beta\gamma\iota) \qquad (h, \sigma)^{00}_{20} = b_a p_\beta q_\theta r_\gamma (acu)(\iota\kappa x)$$
$$(h, \sigma)^{00}_{30} = b_a p_\beta q_\theta r_\gamma (ac\,\overline{\iota\kappa}) \qquad (h, \tau)^{00}_{03} = b_a q_\theta (\beta\iota x)(\gamma\kappa x)(\overline{ac}\,\overline{pr}$$
$$(f^2, \phi)^{11}_{00} = a_\theta a_u \beta_u (bpu) \qquad (f^2, \phi)^{00}_{11} = p_a a_x b_x (\beta\theta x)$$
$$(f^2, \phi_1)^{11}_{00} = a_\iota q_\theta a_u \beta_u (bpu) \qquad (f^2, \phi_1)^{00}_{11} = p_a q_\theta a_x b_x (\beta\iota x)$$
$$(ff_1, \phi)^{11}_{00} = a_\theta c_\beta a_u \gamma_u (bpu) \qquad (ff_1, \phi)^{00}_{11} = c_\beta p_a a_x b_x (\gamma\theta x)$$
$$(ff_1, \phi_1)^{11}_{00} = a_\iota c_\beta q_\theta a_u \gamma_u (bpu) \qquad (ff_1, \phi_1)^{00}_{11} = c_\beta p_a q_\theta a_x b_x (\gamma\iota x)$$
$$(f_1^2, \phi)^{11}_{00} = a_\theta b_a d_\gamma \beta_u \delta_u (cpu) \qquad (f_1^2, \phi)^{00}_{11} = b_a d_\gamma p_\beta a_x c_x (\delta\theta x)$$
$$(f_1^2, \phi_1)^{11}_{00} = a_\iota b_a d_\gamma q_\theta \beta_u \delta_u (cpu) \qquad (f_1^2, \phi_1)^{00}_{11} = b_a d_\gamma p_\beta q_\theta a_x c_x (\delta\iota x)$$

GRADE 7

$$(g,\phi^2)^{\,20}_{\,00} = a_\theta b_a c_\iota p_x q_x (\beta\gamma x) \qquad (g,\phi^2)^{\,21}_{\,00} = a_\theta b_a c_\iota (\overline{\beta\gamma} pu) q_x$$

$$(g,\phi^2)^{\,12}_{\,00} = a_\theta b_{a\iota u} (cpu)(\overline{\beta\gamma} qu) \qquad (g,\phi\phi_1)^{\,20}_{\,00} = a_\theta b_a r_\iota c_\kappa p_x q_x (\beta\gamma x)$$

$$(g,\phi\phi_1)^{\,21}_{\,00} = a_\theta b_a c_\kappa r_\iota (\overline{\beta\gamma} pu) q_x \qquad (g,\phi\phi_1)^{\,12}_{\,00} = a_\theta b_a r_{\iota\kappa u} (cpu)(\overline{\beta\gamma} qu)$$

$$(g,\phi_1^2)^{\,20}_{\,00} = a_\iota b_a c_\lambda q_\theta s_\kappa p_x r_x (\beta\gamma x) \qquad (g,\phi_1^2)^{\,21}_{\,00} = a_\iota b_a c_\lambda q_\theta s_\kappa (\overline{\beta\gamma} pu) r_x$$

$$(g,\phi_1^2)^{\,12}_{\,00} = a_\iota b_a q_\theta s_\kappa \lambda_u (cpu)(\overline{\beta\gamma} ru) \qquad (h,\phi^2)^{\,00}_{\,20} = b_a p_\beta q_\gamma (acu)\theta_{u\iota u}$$

$$(h,\phi^2)^{\,00}_{\,21} = b_a p_\beta q_\gamma (\overline{ac\theta} x)\iota_u \qquad (h,\phi^2)^{\,00}_{\,12} = b_a p_\beta (\gamma\theta x)(\overline{ac\iota} x) q_x$$

$$(h,\phi\phi_1)^{\,00}_{\,20} = b_a p_\beta q_\gamma r_\iota \theta_u \kappa_u (acu)$$

$$(h,\phi\phi_1)^{\,00}_{\,21} = b_a p_\beta q_\gamma r_\iota (\overline{ac\theta} x)\kappa_u$$

$$(h,\phi\phi_1)^{\,00}_{\,12} = b_a p_\beta r_\iota q_x (\gamma\theta x)(\overline{ac\kappa} x)$$

$$(h,\phi_1^2)^{\,00}_{\,20} = b_a p_\beta q_\theta r_\gamma s_{\kappa\iota u} \lambda_u (acu)$$

$$(h,\phi_1^2)^{\,00}_{\,21} = b_a p_\beta q_\theta r_\gamma s_\kappa \lambda_u (\overline{ac\iota} x)$$

$$(h,\phi_1^2)^{\,00}_{\,12} = b_a p_\beta q_\theta s_\kappa r_x (\gamma\iota x)(\overline{ac\lambda} x)$$

$$(f^2,\sigma)^{\,00}_{\,20} = q_\theta p_a r_\beta a_x b_x (\iota\kappa x) \qquad (f^2,\sigma)^{\,01}_{\,20} = q_\theta a_r (\beta\iota\kappa)(apu) b_x$$

$$(f^2,\sigma)^{\,02}_{\,10} = q_\theta (a\iota\kappa)\beta_u (apu)(bru) \qquad (ff_1,\sigma)^{\,00}_{\,20} = c_\beta p_a q_\theta r_\gamma a_x b_x (\iota\kappa x)$$

$$(ff_1,\sigma)^{\,01}_{\,20} = c_\beta q_\theta r_a (\gamma\iota\kappa)(apu) b_x \qquad (ff_1,\sigma)^{\,02}_{\,10} = c_\beta q_\theta (a\iota\kappa)\gamma_u (apu)(bru)$$

$$(f_1^2,\sigma)^{\,00}_{\,20} = b_a d_\gamma p_\beta q_\theta r_\delta a_x c_x (\iota\kappa x) \qquad (f_1^2,\sigma)^{\,01}_{\,20} = b_a d_\gamma q_\theta r_\beta (\delta\iota\kappa)(apu) c_x$$

$$(f_1^2,\sigma)^{\,02}_{\,10} = b_a d_\gamma q_\theta (\beta\iota\kappa)\delta_u (apu)(cru) \qquad (f^2,\tau)^{\,20}_{\,00} = a_\iota b_\kappa q_\theta a_u \beta_u (pru)$$

$$(f^2,\tau)^{\,20}_{\,01} = a_\iota b_\kappa q_\theta (a\,\overline{pr}\,x)\beta_u \qquad (f^2,\tau)^{\,10}_{\,02} = a_\iota q_\theta b_x (\beta\,\overline{pr}\,x)(a\kappa x)$$

$$(ff_1,\tau)^{\,20}_{\,00} = a_\iota b_\kappa c_\beta q_\theta a_u \gamma_u (pru)$$

$$(ff_1,\tau)^{\,20}_{\,01} = a_\iota b_\kappa c_\beta q_\theta (a\,\overline{pr}\,x)\gamma_u$$

$$(ff_1,\tau)^{\,10}_{\,02} = c_\beta q_\theta (apr) b_x (a\iota x)(\gamma\kappa x)$$

$$(f_1^2,\tau)^{\,20}_{\,00} = a_\iota b_a c_\kappa d_\gamma q_\theta \beta_u \delta_u (pru)$$

$$(f_1^2,\tau)^{\,20}_{\,01} = a_\iota b_a c_\kappa d_\gamma q_\theta (\beta\,\overline{pr}\,x)\delta_u$$

$$(f_1^2,\tau)^{\,10}_{\,02} = a_\iota b_a d_\gamma q_\theta c_x (\delta\,\overline{pr}\,x)(\beta\kappa x)$$

GRADE 8

$$(g,\phi\tau)^{\,21}_{\,00} = a_\theta b_a r_\iota c_\kappa \lambda_u (qsu)(\overline{\beta\gamma}\, pu)$$

$$(g,\phi_1\tau)^{\,21}_{\,00} = b_a c_\iota q_\theta s_\kappa (\beta\gamma\lambda)\mu_u (apu)(rtu)$$

$$(h,\phi\sigma)^{\,00}_{\,21} = b_a p_\gamma r_\iota (acq) s_x (\beta\theta x)(\kappa\lambda x)$$

$$(h,\phi_1\sigma)^{\,00}_{\,21} = b_a p_\gamma q_\theta s_\kappa (acr) t_x (\beta\iota x)(\lambda\mu x)$$

$$(fh,\sigma)^{\,01}_{\,20} = b_a p_\beta q_\theta (\iota\kappa\lambda)\delta_u (acu)(dru)$$

$$(f_1 h,\sigma)^{\,01}_{\,20} = b_a e_\delta p_\beta q_\theta (\iota\kappa\gamma)\epsilon_u (acu)(dru)$$

$$(fg,\tau)^{\,20}_{\,01} = a_\iota b_a q_\theta (prc) d_x (\beta\gamma x)(\delta\kappa x)$$

$$(f_1 g,\tau)^{\,20}_{\,01} = a_\iota b_a e_\delta q_\theta (prc) d_x (\beta\gamma x)(\epsilon\kappa x)$$

GRADE 9

$$(g, \phi^2\phi_1)^{30}_{00} = a_\theta b_\alpha c_\iota (\beta\gamma\lambda)\, p_x q_x r_x s_\kappa$$

$$(g, \phi^2\phi_1)^{03}_{00} = b_\alpha s_\kappa \theta_u \iota_u \lambda_u (apu)\,(cqu)\,(\overline{\beta\gamma}\, ru)$$

$$(g, \phi\phi_1^2)^{30}_{00} = a_\theta b_\alpha c_\kappa r_\iota t_\lambda (\beta\gamma\mu)\, p_x q_x s_x$$

$$(g, \phi\phi_1^2)^{03}_{00} = b_\alpha r_\iota t_\lambda \theta_u \kappa_u \mu_u (apu)\,(cqu)\,(\overline{\beta\gamma}\, su)$$

$$(g, \phi^3)^{30}_{00} = a_\theta b_\alpha c_\iota (\beta\gamma\kappa)\, p_x q_x r_x$$

$$(g, \phi^3)^{03}_{00} = b_\alpha \theta_u \iota_u \kappa_u (apu)\,(cqu)\,(\overline{\beta\gamma}\, ru)$$

$$(g, \phi_1^3)^{30}_{00} = a_\iota b_\alpha c_\lambda q_\theta s_\kappa v_\mu (\beta\gamma\nu)\, p_x r_x t_x$$

$$(g, \phi_1^3)^{03}_{00} = b_\alpha q_\theta s_\kappa v_\mu \iota_u \lambda_u \nu_u (apu)\,(cru)\,(\overline{\beta\gamma}\, tu)$$

$$(h, \phi^2\phi_1)^{00}_{30} = b_\alpha p_\beta q_\gamma s_\kappa (acr)\, \theta_u \iota_u \lambda_u$$

$$(h, \phi^2\phi_1)^{00}_{03} = b_\alpha s_\kappa p_x q_x r_x (\beta\theta x)\,(\gamma\iota x)\,(\overline{ac}\, \lambda x)$$

$$(h, \phi\phi_1^2)^{00}_{30} = b_\alpha p_\beta q_\gamma r_\iota t_\lambda (acs)\, \theta_u \kappa_u \mu_u$$

$$(h, \phi\phi_1^2)^{00}_{03} = b_\alpha r_\iota t_\lambda p_x q_x s_x (\beta\theta x)\,(\gamma\kappa x)\,(\overline{ac}\, \mu x)$$

$$(h, \phi^3)^{00}_{30} = b_\alpha p_\beta q_\gamma (acr)\, \theta_u \iota_u \kappa_u$$

$$(h, \phi^3)^{00}_{03} = b_\alpha p_x q_x r_x (\beta\theta x)\,(\gamma\iota x)\,(\overline{ac}\, \kappa x)$$

$$(h, \phi_1^3)^{00}_{30} = b_\alpha p_\beta q_\theta r_\gamma s_\kappa v_\mu (act)\, \iota_u \lambda_u \nu_u$$

$$(h, \phi_1^3)^{00}_{03} = b_\alpha q_\theta s_\kappa v_\mu p_x r_x t_x (\beta\iota x)\,(\gamma\lambda x)\,(\overline{ac}\, \nu x)$$

$$(f^2, \phi\sigma)^{02}_{20} = p_\beta r_\iota (\alpha\kappa\lambda)\, \theta_u (aqu)\,(bsu)$$

$$(f^2, \phi\tau)^{20}_{02} = a_\kappa b_\lambda r_\iota p_x (a\theta x)\,(\beta\, \overline{qs}\, x)$$

$$(f^2, \phi_1\sigma)^{02}_{20} = q_\theta s_\kappa t_\alpha (\beta\lambda\mu)\, \iota_u (apu)\,(bru)$$

$$(f^2, \phi_1\tau)^{20}_{02} = a_\iota b_\mu q_\theta s_\kappa p_x (a\lambda x)\,(\beta\, \overline{rt}\, x)$$

$$(f_1^2, \phi\sigma)^{02}_{20} = b_\alpha d_\gamma r_\iota s_\beta (\delta\kappa\lambda)\, \theta_u (apu)\,(cqu)$$

$$(f_1^2, \phi\tau)^{20}_{02} = a_\theta b_\alpha c_\kappa d_\gamma r_\iota p_x (\beta\lambda x)\,(\delta\, \overline{qs}\, x)$$

$$(f_1^2, \phi_1\sigma)^{02}_{20} = b_\alpha d_\gamma q_\theta s_\kappa t_\beta (\delta\lambda\mu)\, \iota_u (apu)\,(cru)$$

$$(f_1^2, \phi_1\tau)^{20}_{02} = a_\iota b_\alpha c_\lambda d_\gamma q_\theta s_\kappa p_x (\beta\mu x)\,(\delta\, \overline{rt}\, x)$$

$$(ff_1, \phi\sigma)^{02}_{20} = c_\beta r_\iota s_\alpha (\gamma\kappa\lambda)\, \theta_u (apu)\,(bqu)$$

$$(ff_1, \phi\tau)^{20}_{02} = a_\theta b_\kappa c_\beta r_\iota p_x (a\lambda x)\,(\gamma\, \overline{qs}\, x)$$

$$(ff_1, \phi_1\sigma)^{02}_{20} = c_\beta q_\theta s_\kappa t_\alpha (\gamma\lambda\mu)\, \iota_u (apu)\,(bru)$$

$$(ff_1, \phi_1\tau)^{20}_{02} = a_\iota b_\lambda c_\beta q_\theta s_\kappa p_x (a\mu x)\,(\gamma\, \overline{rt}\, x)$$

$$(f^2 f_1, \sigma)^{00}_{30} = b_\alpha p_\beta q_\theta r_\gamma (\iota\kappa\delta)\, a_x c_x d_x$$

$$(f^2 f_1, \sigma)^{03}_{00} = b_\alpha q_\theta \beta_u \gamma_u \delta_u (apu)\,(cru)\,(d\, \overline{\iota\kappa}\, u)$$

$$(ff_1^2, \sigma)_{30}^{00} = b_a d_\gamma p_\beta q_\theta r_\delta (\epsilon\iota\kappa) a_x c_x e_x$$

$$(ff_1^2, \sigma)_{00}^{03} = b_a d_\gamma q_\theta \beta_u \delta_u \epsilon_u (apu)(cru)(e\ \overline{\iota\kappa}\ u)$$

$$(f^3, \sigma)_{30}^{00} = q_\theta p_a r_\beta (\iota\kappa\gamma) a_x b_x c_x$$

$$(f^3, \sigma)_{00}^{03} = q_\theta a_u \beta_u \gamma_u (apu)(bru)(c\ \overline{\iota\kappa}\ u)$$

$$(f_1^3, \sigma)_{30}^{00} = b_a d_\gamma f_\epsilon p_\beta q_\theta r_\delta (\iota\kappa\zeta) a_x c_x e_x$$

$$(f_1^3, \sigma)_{00}^{03} = b_a d_\gamma f_\epsilon q_\theta \beta_u \delta_u \zeta_u (apu)(cru)(e\ \overline{\iota\kappa}\ u)$$

$$(f^2 f_1, \tau)_{00}^{30} = a_\iota b_a c_\kappa q_\theta (prd) \beta_u \gamma_u \delta_u$$

$$(f^2 f_1, \tau)_{03}^{00} = b_a q_\theta a_x c_x d_x (\beta\iota x)(\gamma\kappa x)(\delta\ \overline{pr}\ x)$$

$$(ff_1^2, \tau)_{00}^{30} = a_\iota b_a c_\kappa d_\gamma q_\theta (pre) \beta_u \delta_u \epsilon_u$$

$$(ff_1^2, \tau)_{03}^{00} = b_a d_\gamma q_\theta a_x c_x e_x (\beta\iota x)(\delta\kappa x)(\epsilon\ \overline{pr}\ x)$$

$$(f^3, \tau)_{00}^{30} = a_\iota b_\kappa q_\theta (prc) a_u \beta_u \gamma_u$$

$$(f^3, \tau)_{03}^{00} = q_\theta a_x b_x c_x (a\iota x)(\beta\kappa x)(\gamma\ \overline{pr}\ x)$$

$$(f_1^3, \tau)_{00}^{30} = a_\iota b_a c_\kappa d_\gamma f_\epsilon q_\theta (pre) \beta_u \delta_u \zeta_u$$

$$(f_1^3, \tau)_{03}^{00} = b_a d_\gamma f_\epsilon q_\theta a_x c_x e_x (\beta\iota x)(\delta\kappa x)(\zeta\ \overline{pr}\ x)$$

$$(fg, \phi^2)_{00}^{22} = a_\theta c_\beta (\gamma\delta\iota) a_u (bpu)(dqu)$$

$$(fh, \phi^2)_{22}^{00} = c_\beta p_a (bdq) a_x (\gamma\theta x)(\delta\iota x)$$

$$(f_1 g, \phi^2)_{00}^{22} = a_\theta b_a d_\gamma (\delta\epsilon\iota) \beta_u (cpu)(equ)$$

$$(f_1 h, \phi^2)_{22}^{00} = b_a d_\gamma p_\beta (ceq) a_x (\delta\theta x)(\epsilon\iota x)$$

$$(fg, \phi_1^2)_{00}^{22} = a_\iota c_\beta q_\theta s_\kappa (\gamma\delta\lambda) a_u (bpu)\ (dru)$$

$$(fh, \phi_1^2)_{22}^{00} = c_\beta p_a q_\theta s_\kappa (bdr) a_x (\gamma\iota x)(\delta\lambda x)$$

$$(f_1 g, \phi_1^2)_{00}^{22} = a_\iota b_a d_\gamma q_\theta s_\kappa (\delta\epsilon\lambda) \beta_u (cpu)(eru)$$

$$(f_1 h, \phi_1^2)_{22}^{00} = b_a d_\gamma p_\beta q_\theta s_\kappa (cer) a_x (\delta\iota x)(\epsilon\lambda x)$$

$$(fg, \phi\phi_1)_{00}^{22} = a_\iota c_\beta q_\theta (\gamma\delta\kappa) a_u (bpu)(dru)$$

$$(fh, \phi\phi_1)_{22}^{00} = c_\beta p_a q_\theta (bdr) a_x (\gamma\iota x)(\delta\kappa x)$$

$$(f_1 g, \phi\phi_1)_{00}^{22} = a_\iota b_a d_\gamma q_\theta (\delta\epsilon\kappa) \beta_u (cpu)(eru)$$

$$(f_1 h, \phi\phi_1)_{22}^{00} = b_a d_\gamma p_\beta q_\theta (cer) a_x (\delta\iota x)(\epsilon\kappa x)$$

GRADES 10, 11

No irreducible forms.

GRADE 12

$$(f^3, \sigma^2)^{03}_{30} = p_a q_\theta r_\beta t_\lambda (\iota\kappa\gamma)(asu)(b\nu u)(c\,\overline{\mu\nu}\,u)$$

$$(f^3, \tau^2)^{30}_{03} = a_\iota b_\kappa q_\theta t_\lambda (cpr)(a\mu x)(\beta\nu x)(\gamma\,\overline{s\upsilon}\,x)$$

$$(f_1{}^3, \sigma^2)^{03}_{30} = b_a d_\gamma f_\epsilon p_\beta q_\theta r_\delta t_\lambda (\iota\kappa\zeta)(asu)(cvu)(e\,\overline{\mu\nu}\,u)$$

$$(f_1{}^3, \tau^2)^{30}_{03} = a_\iota b_a c_\kappa d_\gamma f_\epsilon q_\theta t_\lambda (pre)(\beta\mu x)(\delta\nu x)(\zeta\,\overline{s\upsilon}\,x)$$

$$(f_1 f^2, \sigma^2)^{03}_{30} = b_a p_\beta q_\theta r_\gamma t_\lambda (\delta\iota\kappa)(asu)(cvu)(d\,\overline{\mu\nu}\,u)$$

$$(f_1 f^2, \tau^2)^{30}_{03} = a_\iota b_a c_\kappa q_\theta t_\lambda (dpr)(\beta\mu x)(\gamma\nu x)(\delta\,\overline{s\upsilon}\,x)$$

$$(f_1{}^2 f, \sigma^2)^{03}_{30} = b_a d_\gamma p_\beta q_\theta r_\delta t_\lambda (\epsilon\iota\kappa)(asu)(cvu)(e\,\overline{\mu\nu}\,u)$$

$$(f_1{}^2 f, \tau^2)^{30}_{03} = a_\iota b_a c_\kappa d_\gamma q_\theta t_\lambda (pre)(\beta\mu x)(\delta\nu x)(\epsilon\,\overline{s\upsilon}\,x)$$

$$(g^2, \phi^3)^{33}_{00} = a_\theta b_a c_\iota e_\delta (\beta\gamma\kappa)(dpu)(fqu)(\overline{\epsilon\zeta}\,ru)$$

$$(g^2, \phi_1{}^3)^{33}_{00} = a_\iota b_a c_\lambda e_\delta q_\theta s_\kappa v_\mu (\beta\gamma\nu)(dpu)(fru)(\overline{\epsilon\zeta}\,tu)$$

$$(g^2, \phi_1\phi^2)^{33}_{00} = a_\iota b_a c_\kappa e_\delta q_\theta (\beta\gamma\lambda)(dpu)(fru)(\overline{\epsilon\zeta}\,su)$$

$$(g^2, \phi_1{}^2\phi)^{33}_{00} = a_\iota b_a c_\lambda e_\delta q_\theta s_\kappa (\beta\gamma\mu)(dpu)(fru)(\overline{\epsilon\zeta}\,tu)$$

$$(h^2, \phi^3)^{00}_{33} = b_a e_\delta p_\beta q_\gamma (acr)(\epsilon\theta x)(\zeta\iota x)(\overline{df}\,\kappa x)$$

$$(h^2, \phi_1{}^2\phi)^{00}_{33} = b_a e_\delta p_\beta q_\theta r_\gamma s_\kappa (act)(\epsilon\iota x)(\zeta\lambda x)(\overline{df}\,\mu x)$$

$$(h^2, \phi_1\phi^2)^{00}_{33} = b_a e_\delta p_\beta q_\theta r_\gamma (acs)(\epsilon\iota x)(\zeta\kappa x)(\overline{df}\,\lambda x)$$

$$(h^2, \phi_1{}^3)^{00}_{33} = b_a e_\delta p_\beta q_\theta s_\kappa v_\mu r_\gamma (act)(\epsilon\iota x)(\zeta\lambda x)(\overline{df}\,\nu x)$$

THE UNIVERSITY OF PENNSYLVANIA.

Arithmetics of Generalized Quaternion Algebras.

By Claiborne G. Latimer.

Introduction. We shall consider the algebra, over the field of rational numbers with a modulus, designated as 1, which shall be taken as one of the units, and the additional units I, J, K, with the following multiplication table, α and β being ordinary integers.

$$I^2 = \alpha,\ J^2 = -\beta,\ K^2 = \alpha\beta,\ IJ = -JI = K,$$
$$IK = -KI = \alpha J,\ JK = -KJ = \beta I.$$

A definition of *integral element* will be made, similar to that given by Dickson in his *Algebras and Their Arithmetics.** For α and β odd or $\alpha \equiv 2$, $\beta \equiv 1 \pmod 8$, it will be found that the algebra contains 2^i sets of such elements, where i depends on the number of prime factors of certain character contained in α and β.†

Section 1. *Let* $X = \lambda + \mu I + \xi J + \eta K$ be an element of the algebra. Define the conjugate of X as $\overline{X} = 2\lambda - X$, and the norm of X as $N(X) = \overline{X}X$. Then X is a root of

$$Z^2 - 2\lambda Z + N(X) = 0. \tag{1}$$

Let an *integral element* be an element that belongs to a set, S, with the following properties.

R: For every element of S, the coefficients of (1) are rational integers.
C: S is closed under addition, subtraction and multiplication.
U: S contains the units, 1, I, J, K.
M: S is maximal; i. e. it is not contained in a larger set with the properties R, C, U.

An element will be said to have the property R, if the coefficients of (1) are ordinary integers.

* See pp. 141, 2, 6.

† Similar results when α and β are even are contained in the Chicago thesis of Marguerite Darkow.

Let x^2 and y^2 be the largest square factors of a and β respectively. Make the transformation of units;

$$I = xI' \qquad J = yJ' \qquad K = xyK'.$$

The multiplication table of the new units is the same as that of the original units, except that all square factors greater than unity have been removed from a and β. It will be assumed hereafter that this transformation has been made.

If an element, X, belongs to a set with the properties R, C, U, then IX, JX, KX also belong to the set. Hence the following are ordinary integers.

$$2\lambda = u, \qquad 2a\mu = v, \qquad 2\beta\xi = s, \qquad 2a\beta\eta = t.$$

Therefore every element that belongs to a set with the properties R, C, U may be written in the form:

$$X = \tfrac{1}{2}[u + a^{-1}vI + \beta^{-1}sJ + (a\beta)^{-1}tK] \tag{2}$$

where u, v, s, t are ordinary integers. By the property R, $N(X)$ is a rational integer. Hence we have

THEOREM I. *A necessary condition that an element* (2) *belong to a set with the properties R, C, U, is that u, v, s, t be ordinary integers such that*

$$a\beta u^2 - \beta v^2 + as^2 - t^2 \equiv 0 \pmod{4a\beta} \tag{3}$$

This is a sufficient condition that the element have the property R.

Section 2. Suppose a and β are odd. Then (3) is equivalent to

$$\beta v^2 - as^2 + t^2 \equiv 0 \pmod{a\beta}, \tag{3'}$$

$$a\beta u^2 - \beta v^2 + as^2 - t^2 \equiv 0 \pmod{4}. \tag{3''}$$

Let $a = a_1\delta$, $\beta = \beta_1\delta$, where δ is the g. c. d. of a and β. No two of a_1, β_1, δ have a factor in common. From (3'), t^2, and hence t is divisible by δ. Set $t = \delta r$. Then (3') is equivalent to

$$\beta_1 v^2 + \delta r^2 \equiv 0 \pmod{a_1}, \qquad a_1 s^2 - \delta r^2 \equiv 0 \pmod{\beta_1}, \qquad \beta_1 v^2 - a_1 s^2 \equiv 0 \pmod{\delta}. \tag{4}$$

Let $a_1 = Aa'$, $\beta_1 = B\beta'$, $\delta = \Delta\delta'$, where $A[B, \Delta]$ is the product of all the prime factors, $A_i[B_i, \Delta_i]$ of $a_1[\beta_1, \delta]$ such that $-\beta[a, a_1\beta_1]$ is a quadratic non-residue of every $A_i[B_i, \Delta_i]$; or $A = 1$ $[B = 1, \Delta = 1]$ if $a_1[\beta_1, \delta]$ contains no such factors. We shall prove the following

LEMMA 1. *If v, s, t are any integers that satisfy (3′), then*

$$v = A\Delta y, \quad s = B\Delta z, \quad t = AB\delta w, \tag{5}$$

where y, z, w are ordinary integers.

Let A' be a prime factor of A. Suppose there exists a $t = \delta r$ not divisible by A'. There exists an integer, a, such that $ar \equiv 1 \pmod{A'}$. Eliminating r between this and the first congruence of (4) with the modulus A' instead of a_1, we obtain $(a\beta_1 v)^2 \equiv -\beta \pmod{A'}$, contrary to hypothesis that $-\beta$ is a quadratic non-residue of every prime factor of A. Then t, and hence by (4), v is divisible by each of the prime factors of A, and therefore by A. In a similar manner it may be shown that v is divisible by Δ, s by Δ and B, and t by B. Since no two of A, B, Δ, δ' have a factor in common, the lemma is proved.

Substitute in (4) ABw for r and the expressions in (5) for v and s. After cancellation we obtain the following, equivalent to (4).

$$\beta'\Delta y^2 + \delta' B w^2 \equiv 0 \pmod{a'}, \quad a'\Delta z^2 - \delta' A w^2 \equiv 0 \pmod{\beta'}, \quad a' B z^2 - \beta' A y^2 \equiv 0 \pmod{\delta'}. \tag{6}$$

By definition of a', $-\beta$ is a quadratic residue of every prime factor of a', and hence it is a quadratic residue of a'.* Similarly, a and $a_1\beta_1$ are quadratic residues of β' and δ' respectively. Then there exist odd integers x_0, y_0, z_0 that are solutions of

$$x^2 + \beta \equiv 0 \pmod{a'}, \quad y^2 - a \equiv 0 \pmod{\beta'}, \quad z^2 - a_1\beta_1 \equiv 0 \pmod{\delta'} \tag{7}$$

respectively. Even integers, r, s, t, and odd integers, ξ, η, ζ may be found such that $x_0 + a'r = \beta\xi$, $y_0 + \beta' s = a\eta$, $z_0 + \delta' t = a_1\beta_1\zeta$. By a well known theorem, there exists an integer h, which we may assume to be odd, such that $h \equiv \xi \pmod{a'}$, $h \equiv \eta \pmod{\beta.}$, $h \equiv \zeta \pmod{\delta'}$. Then βh, ah, $a_1\beta_1 h$ are solutions of the corresponding congruences of (7), and h is prime to $a'\beta'\delta'$.

Multiply each of the congruences of (6) by its leading coefficient. The second coefficients become β, $-a$, $-a_1\beta_1$. Substitute for these $-(\beta h)^2$, $-(ah)^2$, $-(a_1\beta_1 h)^2$ respectively. From the resulting congruences we obtain

$$y^2 \equiv (B\delta' hw)^2 \pmod{a'}, \quad z^2 \equiv (A\delta' hw)^2 \pmod{\beta'}, \quad z^2 \equiv (A\beta' hy)^2 \pmod{\delta'}, \tag{8}$$

which are equivalent to (6).

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* See Tschebyscheff, *Theorie der Congruenzen*, German trans. by Schapira, p. 134.

If the expressions (5) are substituted in (2) and (3″), we obtain

$$X = \tfrac{1}{2}[x + (\alpha'\delta')^{-1}yI + (\beta'\delta')^{-1}zJ + (\alpha'\beta'\delta)^{-1}wK] \tag{9}$$

$$\alpha_1\beta_1x^2 - \beta y^2 + \alpha z^2 - w^2 \equiv 0 \pmod 4 \tag{10}$$

where $x = u$.

If $\alpha' \neq \pm 1$, let α_i' $(i = 1, \cdots, m)$ be the prime factors of α'. Similarly, let n, p be the number of prime factors, β_i', δ_i', in β', δ' respectively. Then, since (8) is equivalent to (3′), we have

THEOREM II. *If α and β are odd integers, neither containing a square factor greater than unity, a necessary condition that an element in the form* (9) *belong to a set with the properties R, C, U is that x, y, z, w be ordinary integers that satisfy* (10) *and one congruence from each of the following $m + n + p = q$ pairs of congruences.*

$$\begin{aligned}
&(a_1)\ y \equiv B\delta'hw \quad (b_1)\ y \equiv -B\delta'hw \pmod{\alpha_i'} \quad (i = 1, \cdots, m),\\
&(a_2)\ z \equiv A\delta'hw \quad (b_2)\ z \equiv -A\delta'hw \pmod{\beta_i'} \quad (i = 1 \cdots, n),\\
&(a_3)\ z \equiv A\beta'hy \quad (b_3)\ z \equiv -A\beta'hy \pmod{\delta_i'} \quad (i = 1, \cdots, p).
\end{aligned} \tag{11}$$

This is a sufficient condition that an element have the property R. βh, αh, $\alpha_1\beta_1 h$ are odd solutions of the corresponding congruences of (7) *and h is prime to $\alpha'\beta'\delta'$.*

Section 3. We shall show in this section that the necessary condition of the last theorem is also sufficient that an element be integral. Consider a system of congruences consisting of one congruence from each of the q pairs of congruences, (11). There are $2^q = t$ such systems. Let them be C_i $(i = 1, 2, \cdots, t)$.

LEMMA 2. *For every system of congruences, C_i, there is an odd integer, H_i, such that C_i is equivalent to*

$$y \equiv B\delta'H_iw \pmod{\alpha'}, \quad z \equiv A\delta'H_iw \pmod{\beta'}, \quad z \equiv A\beta'H_iy \pmod{\delta'}, \tag{12}$$

and also to

$$w \equiv -\beta'\Delta H_iy \pmod{\alpha'}, \quad w \equiv \alpha'\Delta H_iz \pmod{\beta'}, \quad y \equiv \alpha'BH_iz \pmod{\delta'}. \tag{12'}$$

Let C_i be one of the above systems. Suppose it contains congruences

from ($11a_1$) and from ($11b_1$). After proper choice of notation, these congruences may be written,

$$(13)\qquad (a)\ y \equiv B\delta' hw \pmod{\alpha_i'}\ (i=1,\cdots,f),$$
$$(b)\ y \equiv -B\delta' hw \pmod{\alpha_i'}\ (i=f+1,\cdots,m).$$

Let $a=\alpha_1'.\alpha_2'.\cdots.\alpha_f'$, $b=\alpha'_{f+1}.\cdots.\alpha'_m$. Since a and b are relatively prime, there are two integers, r and s, such that $ar+bs=1$. Let E be an integer such that $h=E+2har=-E+2hbs$. Substitute the first [second] of these expressions for h in each of the congruences of (13a) [(13b)]. Then since the α_i' are distinct, (13) is equivalent to $y \equiv B\delta' Ew \pmod{a'}$. If C_i contains all of ($11a_1$) [($11b_1$)] set $E=h$ [$E=-h$].

In a similar manner we may find integers, F and G, such that the congruences of C_i from ($11a_2$), ($11b_2$) and from ($11a_3$), ($11b_3$) are equivalent to $z \equiv A\delta' Fw \pmod{a'}$, $z \equiv A\beta' Gy \pmod{\delta'}$. No two of a', β', δ' have a factor in common. By a well known theorem there is an integer, H_i, which we may assume to be odd, such that $H_i \equiv E \pmod{a'}$, $H_i \equiv F \pmod{\beta'}$, $H_i \equiv G \pmod{\delta'}$. This proves the lemma as to (12). βH_i, αH_i, $\alpha_1\beta_1 H_i$, are solutions of the corresponding congruences of (7). Substitute these solutions in (7). From the resulting congruences and (12), we obtain (12′). This completes the proof of the lemma.

A system of integers, x, y, z, w, satisfies (10) if and only if they satisfy certain of the following conditions, the particular conditions applicable in a given case depending on the residues of α and β, modulo 4.

$$(14)\qquad \begin{array}{llll} (a) & x \equiv z & y \equiv w & \\ (b) & x \equiv w & y \equiv z & \\ (c) & x \equiv y & z \equiv w & \pmod 2 \\ (d) & x \equiv y \equiv z \equiv w & & \end{array}$$

Let $S_i[S_i', S_i'', S_i''']$ be the totality of elements in the form (9) with coordinates, x, y, z, w, that satisfy every congruence of an arbitrarily chosen system, C_i, and also both congruences of (14a) [(14b), (14c), (14d)]. By a tentative process it is found that every element in the form (9) may be written in each of the following equivalent forms. We shall assume that the elements of $S_i[S_i'\ S_i''\ S_i''']$ are written in the form placed opposite each set.

$$(15)\qquad \begin{array}{ll} S_i, & X=a_0+a_1P+a_2Q_i+a_3R_i; \\ S_i', & X=b_0+b_1P'+b_2Q_i'+b_3R_i'; \\ S_i'', & X=c_0+c_1P''+c_2Q_i''+c_3R_i; \\ S_i''', & X=d_0+d_1J+d_2Q_i+d_3R_i: \end{array}$$

where

$$P = \tfrac{1}{2}(1+J),\ Q_i = \delta'^{-1}(I + a_1 H_i J),$$
$$R_i = \tfrac{1}{2}[1 + a'^{-1}BH_iI + \beta'^{-1}AH_iJ + (a'\beta'\delta)^{-1}K],$$
$$P' = \tfrac{1}{2}(1+\delta^{-1}K),\ Q_i' = a'^{-1}(I - H_iK),\ P'' = \tfrac{1}{2}(1+I),$$
$$Q_i'' = \delta'^{-1}(\beta_1 H_i I + J),$$
$$R_i' = \tfrac{1}{2}[1 + \delta'^{-1}BH_iI + (\beta'\delta')^{-1}J + (\beta'\delta')^{-1}H_iK],$$

$$a_3 = w,\quad 2a'a_2 = y - B\delta'H_iw,\quad \beta'\delta'a_1 = z - A\delta'H_iw - 2a_1\beta'H_ia_2,$$
$$2a_0 = x - w - a_1,$$
$$b_3 = z,\quad 2\delta'b_2 = y - a'BH_iz,\quad a'\beta'b_1 = w - a'\Delta H_iz + 2\delta\beta'H_ib_2,$$
$$2b_0 = x - z - b_1,$$
$$c_3 = w,\quad 2\beta'c_2 = z - \delta'AH_iw,\quad a'\delta'c_1 = y - \delta'BH_iw - 2\beta_1a'H_ic_2,$$
$$2c_0 = x - w - c_1,$$
$$d_3 = w,\quad 2a'd_2 = y - \delta'BH_iw,\quad 2\beta'\delta'd_1 = z - \delta'AH_iw - 2a_1\beta'H_id_2,$$
$$2d_0 = x - w.$$

If $\beta \equiv 3 \pmod 4$, the coordinates of S_i satisfy (10). Hence, by Theorem II, S_i has the property R. By Lemma 2 and (14a), the a_i in (15) are ordinary integers. If the units, 1, I, J, K, are written in the form (9), it is readily seen that S_i has the property U. It may also be verified that the elements P, Q_i, R_i and all products of these elements belong to S_i. Then, since every element of S_i may be expressed linearly, with rational integral coefficients, in terms of these elements and 1, S_i has the property C.

In a similar manner it may be found that when a and β satisfy one of the conditions below, the sets placed opposite those conditions have the bases shown in (15) and the properties R, C, U.

$$a \equiv \beta \pmod 4,\quad S_i';$$
$$a \equiv 1 \pmod 4,\quad S_i'';$$
$$a \equiv 3 \quad \beta \equiv 1 \pmod 4,\quad S_i''';$$

Then if a and β are odd, for every system of congruences, C_i, there are two sets of elements with the properties R, C, U, unless $a \equiv 3 \pmod 4$ and $\beta \equiv 1 \pmod 4$, when there is one such set. Hence the algebra contains $2t$ or t sets of elements with these properties. It remains to show that they are maximal.

LEMMA. *A necessary condition that two elements belong to the same set with the properties, R, C, U, is that the coordinates of each element satisfy the same pair of congruences from* (14) *and the congruences of the same system, C_i, from* (11).

Let S be a set with the properties R, C, U. Suppose it contains two elements, X and Y, with coordinates that do not satisfy the congruences of the same system, C_i. By Theorem II, the coordinates of each element satisfy one congruence from each of the q pairs of congruences (11). Then the coordinates of X satisfy one and only one of the congruences of a certain pair in (11) and the coordinates of Y satisfy the second congruence of this pair but do not satisfy the first. The coordinates of $X + Y$ do not satisfy either congruence of the above mentioned pair. Hence, by Theorem II, $X + Y$ does not belong to S, contrary to hypothesis that S is closed. Hence the coordinates of every element of S satisfy all the congruences of one of the systems C_i. In a similar manner it may be shown that the coordinates of every element of S satisfy the same pair of congruences from (14). This proves the lemma.

By the preceding lemma, every set with the properties R, C, U is included in one of the $2t$ or t sets of elements that have been found. Hence these sets are maximal and are the only such sets. We have then

THEOREM III. *If α and β are odd integers, neither containing a square factor greater than unity; if $\alpha = \alpha_1\delta$, $\beta = \beta_1\delta$ where δ is the g. c. d. of α and β; if m $[n, p]$ is the number of prime factors, α_i' $[\beta_i', \delta_i']$ contained in α_1 $[\beta_1, \delta]$ such that $(-\beta/\alpha_i') = 1$ $(i = 1, \cdots, m)$ $[(\alpha/\beta_i') = 1$ $(i = 1, \cdots, n)$, $(\alpha_1\beta_1/\delta_i') = 1$ $(i = 1, \cdots, p)]$ and if $m + n + p = q$: then the algebra contains exactly $2^q = t$ or $2t$ sets of elements with the properties R, C, U, M, according as α and β satisfy or do not satisfy the condition $\alpha \equiv 3$, $\beta \equiv 1 \pmod 4$.**

Section 4. We shall assume in this section that, after the removal of square factors from the parameters, β is odd and $\alpha = 2\lambda$ is even.

Let $\lambda = \lambda_1\delta$, $\beta = \beta_1\delta$ where δ is the g. c. d. of λ and β. Let $\lambda_1 = \Lambda\lambda'$, $\beta_1 = \beta' B$, $\delta = \Delta\delta'$, where $\lambda'[\beta', \delta']$ is the product of all the prime factors, $\lambda_i'[\beta_i', \delta_i']$ of $\lambda_1[\beta_1, \delta_1]$ such that $-\beta[2\lambda, 2\lambda_1\beta_1]$ is a quadratic residue of every $\lambda_i'[\beta_i', \delta_i']$ or $\lambda' = 1$ $[\beta' = 1, \delta' = 1]$ if $\lambda_1[\beta_1, \delta]$ contains no such factors. Then as in Section 2, we may find an odd integer, h, such that βh, $2\lambda h$, $2\lambda_1\beta_1 h$ are solutions of $x^2 + \beta \equiv 0 \pmod{\lambda'}$, $y^2 - 2\lambda \equiv 0 \pmod{\beta'}$, $z^2 - 2\beta_1\lambda_1 \equiv 0 \pmod{\delta'}$, respectively. Let $m[n, p]$ be the number of prime

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** Dickson has obtained the same results for $\alpha = -1$, $\beta \equiv 1, 3 \pmod 4$, $\beta' = \pm 1$; whence $t = 1$ in both cases. See his *Algebras and Their Arithmetics*, p. 192.

factors in $\lambda'[\beta', \delta']$. The following theorem may then be proved by the method of Section 2.

THEOREM II′. *If α and β are integers, neither containing a square factor greater than unity, and if β is odd and $\alpha = 2\lambda$ where λ is an integer, a necessary condition that an element in the form*

$$(9') \qquad X = \tfrac{1}{2}[x + (2\lambda'\delta')^{-1}yI + (\beta'\delta')^{-1}zJ + (2\beta'\lambda'\delta)^{-1}wK]$$

belong to a set with the properties R, C, U, is that the x, y, z, w be ordinary integers that satisfy

$$(10') \qquad 2\lambda\beta x^2 - \beta y^2 + 2\lambda z^2 - w^2 \equiv 0 \pmod 8$$

and one congruence from each of the following $m + n + p = q$ pairs of congruences.

$$(11') \quad \begin{array}{llll} (a_1)\ y \equiv B\delta' hw & (b_1)\ y \equiv -B\delta' hw \pmod{\lambda_i'} & (i = 1, \cdots, m); \\ (a_2)\ z \equiv \Lambda\delta' hw & (b_2)\ z \equiv -\Lambda\delta' hw \pmod{\beta_i'} & (i = 1, \cdots, n); \\ (a_3)\ z \equiv \Lambda\beta' hy & (b_3)\ z \equiv -\Lambda\beta' hy \pmod{\delta_i'} & (i = 1, \cdots, p). \end{array}$$

This is a sufficient condition that an element in the form (9′) have the property R.

Let C_i be a system of congruences, consisting of one from each of the q pairs of congruences (11′). Integers, E_i, F_i, G_i, may be found in the same manner as the integers E, F, G, were found in Section 3, such that the following is true.

LEMMA 2′. *For every system of congruences, C_i, there are integers E_i, F_i, G_i, such that C_i is equivalent to*

$$(12_1) \qquad y \equiv B\delta' E_i w \pmod{\lambda'}, \quad z \equiv \Lambda\delta' F_i w \pmod{\beta'}, \quad z \equiv \Lambda\beta' G_i y \pmod{\delta'};$$

and also to

$$(12_1') \qquad w \equiv -\beta'\Delta E_i y \pmod{\lambda'}, \quad w \equiv 2\lambda'\Delta F_i z \pmod{\beta'}, \quad y \equiv 2\lambda' B G_i y \pmod{\delta'}.$$

Any multiple of $\lambda'[\beta', \delta']$ may be added to $E_i[F_i, G_i]$. It will be assumed hereafter that $B\delta' E_i \equiv \Lambda\delta' F_i \equiv \Lambda\beta' G_i \equiv 1 \pmod 8$. Then if $\lambda \equiv j$, $\beta \equiv k$, it follows that $\Delta\beta' E_i \equiv k$, $\Delta\lambda' F_i \equiv j$, $B\lambda' G_i \equiv jk \pmod 8$.

Suppose $\beta \equiv 1$, $\alpha \equiv 2 \pmod 8$. Then a set of integers, x, y, z, w satisfy (10′) if and only if they satisfy one of the following sets of conditions.

$$(14') \quad \begin{array}{ll} \text{(a)} & x \text{ even}, \ y \equiv z \equiv w \pmod 2, \quad y \equiv w \pmod 4 \text{ if both even}; \\ \text{(b)} & z \text{ even}, \ x \equiv y \equiv w \pmod 2, \quad y \equiv w \pmod 4 \text{ if both even}; \\ \text{(c)} & x, z \text{ odd}, \ y, w \text{ even}, \quad y \equiv w + 2 \pmod 4. \end{array}$$

Let $S_i[S_i']$ be the totality of elements in the form (9′) with coordinates, x, y, z, w, that satisfy every congruence of C_i, one of the conditions (14′), and (16a) [(16b)].

$$(16) \quad \begin{array}{ll} \text{(a)} & y \equiv w \pmod 4 \text{ if } x \text{ s even}, y \text{ and } w \text{ odd}; \\ & y \equiv w + 2 \pmod 4 \text{ if } z \text{ is even}, y \text{ and } w \text{ odd}. \\ \text{(b)} & y \equiv w + 2 \pmod 4 \text{ if } x \text{ is even}, y \text{ and } w \text{ odd}; \\ & y \equiv w \pmod 4 \text{ if } z \text{ is even}, y \text{ and } w \text{ odd}. \end{array}$$

By Theorem II′ and the statement immediately preceding (14′), S_i and S_i' have the property R. By a tentative process it is found that every element of S_i and every element of S_i' may be written as follows.

$$\begin{aligned} S_i: &\quad X = a + bI + cM_i + dN_i; \\ S_i': &\quad X = a' + b'L + c'M_i' + d'N_i'. \end{aligned}$$

where

$$\begin{aligned} &M_i = 1/2\delta'[\beta'\delta' + \beta_1 G_i I + J], \\ &N_i = \tfrac{1}{2}[(2\lambda)^{-1}BE_i I + \beta'^{-1}\Lambda F_i J + (2\lambda'\beta'\delta)^{-1}K], \\ &L = \delta^{-1}K, \quad M_i' = 1/2\beta'[\beta'\delta' + J + F_i K], \\ &N_i' = \tfrac{1}{2}[(2\lambda'\delta')^{-1}I + \delta'^{-1}\Lambda G_i J - (2\lambda'\delta')^{-1}E_i K], \\ &d = w, \quad \beta' c = z - \Lambda\delta' F_i w, \quad 4\lambda'\delta' b = y - B\delta' E_i w - 2\beta_1 G_i \lambda' c, \\ &\qquad\qquad 2a = x - z - \Lambda\delta' F_i w, \\ &d' = y', \quad \delta' c' = z - \Lambda\beta' G_i y, \quad 4\lambda'\beta' b' = w + \beta'\Delta E_i y - 2\delta F_i \lambda' c', \\ &\qquad\qquad 2a' = x - z - \Lambda\beta' G_i y. \end{aligned}$$

When an element of $S_i[S_i']$ is expressed in the above form, by (12_1), $(12_1')$, (14′) and (16) it may be found that the coefficients, $a, b, \cdots, d'$ are ordinary integers. It may be verified that S_i and S_i' have the property U; also that the elements $I, M_i, N_i[L, M_i', N_i']$ and all products of these elements belong to $S_i[S_i']$, which therefore has the property C. There are $2^q = t$ systems of congruences, C_i, for each of which we thus have two sets of elements with the properties R, C, U. By an argument similar to that in Sec-

tion 3, it may be shown that these $2t$ sets are maximal and are the only such sets. We have then

THEOREM III'. *If* $\alpha = 2\lambda \equiv 2$, $\beta \equiv 1 \pmod 8$, *neither* α *nor* β *containing a square factor greater than unity; if* $\lambda = \lambda_1\delta$, $\beta = \beta_1\delta$ *where* δ *is the g. c. d. of* λ *and* β; *if* m $[n, p]$ *is the number of prime factors,* $\lambda_i'[\beta_i', \delta_i']$ *contained in* $\lambda_1[\beta_1, \delta]$ *such that* $(-\beta/\lambda_i') = 1$ $(i = 1, \cdots, m)$ $[(2\lambda/\beta_i') = 1, \cdots, n)$; $(2\lambda_1\beta_1/\delta_i') = 1$ $(i = 1, \cdots, p)]$ *and if* $m + n + p = q$; *then the algebra contains exactly* 2^{q+1} *sets of integral elements.*

TULANE UNIVERSITY.

Conditions under which One of Two Given Closed Linear Point Sets may be Thrown into the Other One by a Continuous Transformation of a Plane into Itself*

By R. L. Moore.

In this paper the following theorem will be established.

Theorem. *If, in a plane S, M and N are two closed and bounded linear point sets and there exists a one to one continuous transformation T, with single valued inverse, which throws M into N then there exists a one to one continuous transformation of S into itself which also has a single valued inverse and which carries M into N in such a way that if X belongs to M then it is carried into $T(X)$.*

In the proof of this theorem use will be made of two lemmas.

Lemma 1. *If q is a straight line and e is a positive number and $M_1, M_2, M_3, \cdots, M_n$ is a finite set of mutually exclusive closed and bounded point sets lying on q and all of diameter less than e then there exists a set of broken lines $b_1, b_2, b_3, \cdots, b_n$ all of diameter less than e, such that (a) if $i \neq j$, b_i has no point in common with b_j, (b) for every i, b_i contains M_i and has, as its end-points, the leftmost and rightmost points of M_i and (c) for every i, b_i consists of a finite number of horizontal, and a finite number of vertical, straight line intervals and no horizontal segment of b_i contains a point which lies directly above another point of b_i.*

Proof. With the help of the Heine-Borel Theorem it can easily be seen that there exists a set H of n mutually exclusive point sets $K_1, K_2, K_3, \cdots, K_n$ such that, for every i, K_i contains M_i and consists of a finite set of intervals of q. Let K denote the sum of the point sets $K_1, K_2, K_3, \cdots, K_n$. Consider q as being a horizontal line running from left to right. Let A_1 denote the leftmost point of K. Let $\bar{K}_1$ denote that point set of the set H which contains A_1. There exists a broken line b_1 of diameter less than e such that (a) b_1 has, in common with q, only the intervals of which $\bar{K}_1$ is composed, (b) b_1 is composed entirely of a finite number of vertical, and a finite number of horizontal, straight line intervals and has, as its ends, the leftmost

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and the rightmost points of $\bar{K}_1$, (c) no point of b_1 lies below q, (d) no horizontal segment of b_1 contains a point directly above another point of b_1.

Now let A_2 denote the leftmost point of $K - \bar{K}_1$ and let $\bar{K}_2$ denote that point set of the set H which contains A_2. There exists a broken line b_2, of diameter less than e, such that (1) b_2 has, in common with q, only the intervals of which $\bar{K}_2$ is composed, (2) b_2 contains no point of $b_1 + K - \bar{K}_2$ and has as endpoints the leftmost and the rightmost points of $\bar{K}_2$ and consists entirely of a finite number of horizontal, and a finite number of vertical, intervals, (3) no horizontal segment of b_2 contains a point directly above another point of b_2, (4) if b_2 contains any points below q then the set of all such points is the sum of a finite set T_2 of segments of the arc b_2, each of these segments being composed of the lower base and the two lateral sides (except for their two upper ends) of a rectangle which is so situated that its upper base lies on q (but does not belong to b_2) and contains, in its interior, just one of the intervals of which $\bar{K}_1$ is composed, but does not contain any point of $K - \bar{K}_1$. Similarly if A_3 denotes the leftmost point of $K - (\bar{K}_1 + \bar{K}_2)$ and $\bar{K}_3$ denotes that point set of the collection H which contains A_3 then there exists a broken line b_3 of diameter less than e such that (1) b_3 has, in common with q, only the intervals of which $\bar{K}_3$ is composed, (2) b_3 contains $\bar{K}_3$, but no point of $b_1 + b_2 + K - \bar{K}_3$, and has, as its endpoints, the leftmost and the rightmost points of $\bar{K}_3$ and consists entirely of a finite number of horizontal, and a finite number of vertical, intervals, (3) no horizontal segment of b_3 contains a point directly above another point of b_3, (4) if b_3 contains any points below q then the set of all such points is the sum of a finite set T_3 of segments (of the arc b_3) each of these segments being composed of the lower base and the two lateral sides (except for their upper ends) of a rectangle which is so situated that its upper base lies on q and contains either one and only one of the intervals of which $\bar{K}_2$ is composed or the projection onto q of the horizontal part of some segment of the set T_2, but does not contain points of more than one point set of the set H.

This process may be continued. It follows that there exists a set of arcs $b_1, b_2, b_3, \cdots, b_n$ satisfying all the conditions required in the statement of Lemma 1.

The following lemma may be easily established.

LEMMA 2. *Suppose that A_1B_1 and A_2B_2 are two simple continuous arcs and M_1 and M_2 are closed point sets lying on A_1B_1 and A_2B_2 respectively and there exists a one to one continuous transformation T with single valued inverse which carries M_1 into M_2. Suppose, furthermore, that if the points X and Y belong to M_1 and X precedes Y on A_1B_1 then $T(X)$ precedes $T(Y)$*

on A_2B_2. Suppose J_1 and J_2 are simple closed curves containing A_1B_1 and A_2B_2 respectively. Then there exists a one to one continuous transformation Z, with single valued inverse, which carries J_1 into J_2 in such a way that if X is any point of M_1 then $Z(X) = T(X)$.

Proof of Theorem. It is clear that the Theorem holds true for the case where M consists of a single point. Suppose it contains more than one point. Then there exists a straight line interval AB which contains M and whose endpoints belong to M. There exists a positive number d_1 such that if X and Y are two points of M at a distance apart less than d_1 then $T(X)$ and $T(Y)$ are at a distance apart less than 1. Let S_1 denote the set of all maximal subintervals of AB (if there be any) which are subsets of M and whose images under T are of length greater than or equal to 1. When the interiors of these intervals are removed from AB the remainder of AB consists of a finite set of intervals. Each of these remaining intervals may be divided into a finite number of nonoverlapping subintervals such that the image, under T, of the point set common to M and any one of these subintervals is of length less than 1 and such that no one of these subintervals has both ends in M and, indeed, neither end of any one of them belongs to M unless it coincides with A or with B or with an end of some interval of the set S_1. Thus the interval AB is divided into four infinite sets S_1, U_1, V_1 and W_1 of nonoverlapping intervals such that (a) S_1 consists of all maximal subintervals of AB which are subsets of M and whose images, under T, are of length greater than or equal to 1, (b) each interval of the set U_1 contains at least one point of M but has no point of M (except possibly A or B) as either of its endpoints, (c) each interval of V_1 has one end in common with some interval of S_1, its other end however not being a point of M, (d) no interval of W_1 contains any point of M, (e) the image, under T, of the point set common to M and any interval belonging to U_1, or to V_1, is of length less than 1. Let C and D denote the ends of the smallest straight line interval which contains N. Let $\bar{S}_1$ denote the set of all those subintervals of CD which are images, under the transformation T, of intervals of the set S_1. If t is any subinterval of AB let M_t denote the set of all those points of M (if there be any) that belong to t. By Lemma 1, if CD is regarded as a horizontal interval, there exists a finite set Q of mutually exclusive simple continuous arcs such that (1) each arc of the set Q is a broken line, of diameter less than 1, consisting entirely of a finite number of horizontal, and a finite number of vertical, straight line intervals, (2) the number of arcs in the set Q is equal to the number of intervals in the set U_1 plus the number of intervals in the set V_1, (3) for each interval t

which belongs to U_1 or to V_1 there exists an arc q_t of the set Q which has, as its ends, the leftmost and the rightmost, points of $T(M_t)$ and which contains $T(M_t)$ but does not contain any point which is the image, under the transformation T, of a point of M belonging to any interval, except t, of the set U_1 or of the set V_1, (4) no horizontal segment of an arc of the set Q contains a point directly above another point of that arc. Since, for each t, the set of points common to q_t and CD is a subset of that subinterval of CD whose ends are the rightmost and leftmost points of $T(M_t)$ and, furthermore, the diameter of $T(M_t)$ is less than 1 and that of every interval of $\bar{S}_1$ is equal to or greater than 1, therefore q_t contains no point of any interval of $\bar{S}_1$ unless t belongs to V_1, in which case q_t has an endpoint in common with one interval of $\bar{S}_1$ but has no other point in common with any interval of $\bar{S}_1$. Thus q_t contains $T(M_t)$ and contains no point of K which does not belong to $T(M_t)$.

It is easy to see that there exist two sets G_1 and G_2 of simple closed curves of diameter less than 1 such that (a) every curve of G_1 or of G_2 is without every other curve belonging to G_1 or to G_2, (b) each interval of $\bar{S}_1$ has one extremity on one, and its other extremity on another, curve of the set G_2 but lies wholly without every curve of G_1 and, except for its extremities, wholly without every curve of G_2, (c) each curve of the set G_2 contains just one point of N, that point being an endpoint of some interval of the set $\bar{S}_1$, (d) for each interval t which belongs to U_1 there exists a curve of the set G_1 which encloses q_t and neither contains nor encloses any point of any interval of $\bar{S}_1$ or any arc except q_t of the set Q, (e) if, for each interval t of the set V_1, A_t denotes the point which is a common endpoint of t and some interval of the set S_1 then there exists a curve of the set G_2 which encloses $q_t - T(A_t)$ and contains $T(A_t)$, (f) each curve of the set G_1 encloses q_t for some interval t of the set U_1 and each curve of the set G_2 contains one endpoint of q_t and encloses the remainder of q_t for some interval t of the set V_1. A curve h belonging to G_1 or to G_2, or an interval h belonging to $\bar{S}_1$, will be said to precede another such curve or interval k provided some point of M whose image under T is enclosed by, or belongs to, h precedes (in the order from A to B on the interval AB) some one whose image under T is enclosed by, or belongs to, k.

Let Σ_1 denote the collection whose elements are the curves of G_1 and of G_2 and the intervals of $\bar{S}_1$. There clearly exists a set H_1 of simple continuous arcs such that (a) if g is any simple closed curve which belongs to Σ_1 and which does not contain any endpoint of any interval of $\bar{S}_1$ which follows g and g is not the last curve of Σ_1 then there is an arc of H_1 whose endpoints lie respectively on g and the first curve of Σ_1 which follows g, (b) every arc of H_1 has its endpoints on two simple closed curves g and h of Σ_1

such that h is the first element of Σ_1 which follows g, (c) no two arcs of H_1 have a point in common, no one of them contains a point of any interval of $\bar{S}_1$ and no one of them has any point, except its endpoints, in common with any curve of Σ_1. There exists an arc-curve chain * K_1 whose curves are the curves of G_1 and of G_2 and whose arcs are the arcs of H_1 and of $\bar{S}_1$. Every arc of the chain K_1 either contains no point of N or is a maximal connected subset of N. Every curve of K_1 is of diameter less than 1. No curve of K_1 contains more than one point of N and if it does contain a point of N that point is an endpoint of an arc of K_1 which consists entirely of points of N. The arc-curve chain K_1 covers N. Furthermore if X and Y are distinct points of M and X precedes Y on the interval AB then either $T(X)$ and $T(Y)$ are interior to the same arc or within the same curve of K_1 or there is an arc or curve of K_1 which contains or encloses $T(X)$ and which precedes one which contains or encloses $T(Y)$. In a similar way it may be shown that there exists an arc-curve chain K_2, covering N, such that (a) every curve of K_2 is of diameter less than $\frac{1}{2}$, (b) the interior of every curve of K_2 is within some curve of K_1 and every point which belongs to an arc of K_2 either belongs to an arc of K_1 or lies within a curve of K_1, (c) no curve of K_2 has a point in common with any curve of K_1 unless that point is an endpoint of some interval of the set $\bar{S}_1$, (d) every arc of K_1 is a subset of some arc of K_2 and every arc of K_2 either contains no point of N or is a maximal connected subset of N, (e) if X and Y are points of M such that $T(X)$ and $T(Y)$ are not interior to the same arc, or within the same curve, of K_2 then X precedes Y on AB if, and only if, some arc or curve of K_2 that contains or encloses $T(X)$ precedes one that contains or encloses $T(Y)$. This process may be continued. Thus we have a sequence of arc-curve chains $K_1, K_2, K_3, \cdots$ such that (a) for every n, K_n covers N, (b) for every n, the interior of every curve of K_{n+1} is within some curve of K_n, (c) no curve of K_{n+1} has a point in common with any curve of K_n unless that point is an endpoint of a maximal arc belonging to the point set N, (c) every arc of K_n is a subset of some

* An *arc-curve* chain is a finite set K of simple closed curves $a_1, a_2, a_3, \ldots, a_n$ and arcs $A_1B_1, A_2B_2, A_3B_3, \ldots, A_{n-1}B_{n-1}$ such that (1) every curve of K is without every other curve of K, (2) no two arcs of K have a point in common, (3) for every i $(1 \leqq i \leqq n-1)$ the arc A_iB_i lies entirely without every curve of K except that A_i is on a_i and B_i is on a_{i+1}. The curve a_i is said to *precede* the curve a_j if $i < j$. The curve a_1 is said to be the *first*, and a_n is said to be the *last*, curve of the arc-curve chain K. The arc-curve chain K is said to *cover* the point set M if every point of M is either within a curve, or on an arc, of K. Cf. R. L. Moore and J. R. Kline, "On the most general plane closed point set through which it is possible to pass a simple continuous arc," *Annals of Mathematics*, Vol. 20 (1919), pp. 218-223.

arc of K_{n+1} and every arc of K_{n+1} either contains no point of N or is a maximal connected subset of N, (d) every point which belongs to an arc of K_{n+1} either belongs to an arc of K_n or lies within a curve of K_n, (e) every curve of K_n is of diameter less than $1/n$, (f) if X and Y are points of M such that $T(X)$ and $T(Y)$ are not interior to the same arc, or within the same curve, of K_n then X precedes Y on AB if, and only if, some arc or curve of K_n that contains or encloses $T(X)$ precedes one that contains or encloses $T(Y)$.

For every n, let $\bar{K}_n$ denote the set of all points $[X]$ such that X belongs to an arc or a curve, or is within a curve, of the chain K_n. Let a denote the set of all points common to the point sets of the sequence $\bar{K}_1, \bar{K}_2, \bar{K}_3, \cdots$. Let $\overline{A}$ and $\overline{B}$ denote the points $T(A)$ and $T(B)$ respectively. With the use of methods largely similar to those employed in the proof of Theorem 15 of my paper "On the Foundations of Plane Analysis Situs," * it can be shown that a is a simple continuous arc with endpoints at $\overline{A}$ and $\overline{B}$. It is clear that this arc contains every point of N and that if X and Y are two points of M then X precedes Y on AB if, and only if, $T(X)$ precedes $T(Y)$, on the arc a, in the order from $\overline{A}$ to $\overline{B}$. Let J and $\overline{J}$ denote two simple closed curves containing the straight line interval AB and the arc a respectively. By Lemma 2, there exists a one to one continuous transformation Z, with single valued inverse, which carries J into $\overline{J}$ and which is such that if X is a point of M then $Z(X) = T(X)$. By a well known theorem due to Schoenflies, there exists a continuous one to one transformation W, with single valued inverse, which throws S into itself and which is such that if X is any point of J then $W(X) = Z(X)$. If X belongs to M then $Z(X) = T(X)$ and therefore $W(X) = T(X)$. The truth of the Theorem is therefore established.

* *Transactions of the American Mathematical Society*, Vol. 17 (1916), pp. 131-164.